ROBUST MULTI-PRODUCT NEWSVENDOR PROBLEM UNDER A GLOBAL BUDGET OF UNCERTAINTY

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

James Dong

May 2018

ROBUST MULTI-PRODUCT NEWSVENDOR PROBLEM UNDER A GLOBAL BUDGET OF UNCERTAINTY

James Dong, Ph.D.

Cornell University 2018

We consider a single-location, single-period stock allocation problem (newsvendor-like problem) with n items in which demand rates, holding costs, and backorder costs vary across all products. Inventory levels are replenished at the end of each period instantaneously. We apply robust optimization under an uncertainty set that captures a risk pooling phenomenon across items to this problem. The number of constraints governing the uncertainty set grows linearly in the number of items. A closed form solution is presented for the single and two-item cases. For the general n item problem, we present a 2-approximation algorithm and demonstrate its asymptotic optimality. The experimental results confirm the value of the approximation algorithm and indicate that the average performance is close to optimal.

BIOGRAPHICAL SKETCH

James Dong was born on June 29, 1991 in Bloomington, Indiana. He then found his way to Georgia where he completed his secondary education. He enrolled at Columbia University somewhat against his parents' wishes and completed his B.S. in Operations Research in the Spring of 2013. James began his Ph.D at Cornell later that Fall.

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my parents, Jianping and Zhuan, for their support and the sacrifices they have made for my education. I would also like to express my deep gratitude to my advisory committee of Professors Jack Muckstadt, Peter Jackson, and Jamol Pender, whose patience and support have made this endeavor possible. Their considerable efforts in my time here have resulted in an extensive, well-rounded education in both theory and practice and both as an educator and as a student. I trust that many more students will benefit from their guidance in the years to come.

I also owe a debt of gratitude to Professor Ward Whitt, whose Service Engineering class inspired me to begin this journey. Benefiting from his guidance in my early career and his tireless curiosity has helped to shape me into who I am today.

I would like to thank the faculty of the School of ORIE for the privilege of learning from world-renowned researchers and for creating a collaborative environment for students to learn and grow. I would like to especially thank Professor Huseyin Topaloglu for the direction he provided in my time here. Additionally, I wish to express my thanks to the staff of the School of ORIE for their many kindnesses over the past five years.

Finally, I am grateful to the many Ph.D students within the School of ORIE who have made Cornell such a wonderful place. I would like to especially thank Kenneth Chong, Ravi Kumar, and Steve Pallone for the many hours of thoughtful discussion.

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1 Introduction

In this paper we study a single period, single location, multi-item newsvendor-like problem when demand is captured through an uncertainty set and stock levels are determined for each item using robust optimization.

The newsvendor problem is a simple but rich inventory model which has been widely studied due to its versatility and applicability to many business decision problems in fields such as travel, health care, scheduling, and others. In the classic newsvendor problem, the seller of an inventory of homogeneous items attempts to maximize expected profit by setting inventory levels in the face of random demand from a known distribution before the exact demand is realized. There are two modeling perspectives on the newsvendor problem - one is as a recourse problem and the other is as a single period version of the newsvendor problem. In the recourse model, inventory levels are chosen in the first period, then after the realization of demand, a course of action must be taken in the second at some cost to the seller, i.e. backordered demand is either met through a third-party at higher cost or perhaps a cost in the form of the loss of goodwill is incurred. In the single period version of the newsvendor problem, costs associated with the realized demand are incurred in the same period.

However, in some settings the traditional assumption that the demand distribution is known may be unrealistic. For example, the observed history may be too short or too variable to reliably estimate the distribution parameters or even the type of distribution. This may arise in the case of newly launched products or in the case of several large customers whose demands vary substantially over a replenishment lead time and who represent a large share of sales. To overcome this problem, distribution-free approaches have been considered in the literature. Scarf (1998) [21] was the first to give a closed-form solution to the newsvendor problem when only the demand mean and variance are known. Scarf's approach identifies the worst possible distribution among those sharing the first two moments as a distribution with positive mass at two points and optimizes with respect to this distribution. Gallego and Moon (1993) [13] were able to provide a simpler proof of Scarf's closed form solution as well as to extend the main idea to the cases with recourse, fixed ordering cost, random yields, and multiple items.

While Scarf's approach yields a closed form solution that is useful for managerial insight, the solution is quite conservative. This "price of robustness" is not unique to Scarf's approach, but a result of the maximin approach which seeks to maximize the worst-case profit. In other words, the planner accepts a suboptimal solution for the original data/parameters of the model in order to ensure that the solution remains feasible and near optimal for perturbations in the data. Several papers have attempted to mitigate the conservativeness of this worst-case approach by designing uncertainty sets to capture either the demand or the demand distribution and by optimizing with respect to demand that can arise from the uncertainty set or a distribution within the uncertainty set.

In references [20] and [13] where the seller is choosing an inventory level that is robust to the uncertainty in the demand distributions from the set of distributions with a fixed first and second moment. However, robust optimization can also be used to think of uncertainty in other parameters such as budgets or costs.

One of the earliest papers on robust optimization was presented by Soyster (1973) [22] who sought to find a solution that would be feasible among small perturbations in the data when the perturbations were drawn from a polytope defined by independent constraints. This work was then extended by Ben-Tal and Nemirovski (1999, 2000) [2, 3], El Ghaoui and Lebret (1997) [11], and El Ghaoui et al [12] who modeled uncertainty with ellipsoidal sets, reducing the conservativeness of the solutions. However, this approach results in nonlinear convex optimization problems, which are more computationally challenging than the original linear program. Most recently, Bertsimas and Sim (2003, 2004) [6, 7] show how to efficiently solve robust optimization problems with polyhedral uncertainty sets. The approach in reference [6] utilizes the idea of a "budget" of uncertainty that can be thought of as a parameter utilized by the planner to choose the desired trade off between robustness and conservativeness. Bertsimas et al (2011) [4] illustrate how this idea can be used to model many uncertainty sets under a unified framework.

Bertsimas and Thiele (2006) [8] apply the approach found in reference [7] to inventory theory, beginning with the newsvendor problem and extending their analysis to include tree networks with capacity constraints on inventory at each node. The uncertainty set utilized by Bertsimas and Thiele is quite similar to the one used in this paper, and bounds the total scaled deviation of demand. Supposing that demand for a single item in period k, w_k, takes values in $[\bar{w}_k - \hat{w}_k, \bar{w}_k + \hat{w}_k]$, let the scaled deviation be represented by $z_k = (w_k - \bar{w}_k)/\hat{w}_k$. Then z_k represents how much a particular demand value deviates from its mean, normalized by the bounds of deviation. The uncertainty set used fixes a budget in each period k to be Γ_k, and bounds the total deviation with $\sum_{i=0}^{k} |z_i| \leq \Gamma_k \ \forall \ k$. The constraints applied to the deviations are termed "budgets of uncertainty" and prevent large deviations in cumulative demand, and can be thought of as preventing an unreasonable worst-case demand scenario.

Terry et al (2010) [23] utilize the framework found in reference [8] to solve a very similar problem to the one studied in this paper. In fact, they study two-stage robust optimization models with recourse which encompass the newsvendor-like model studied in this paper. In their approach, they are able to solve large problems to optimality through a cutting planes algorithm, which cannot be done in our problem formulation. In particular, their model differs from ours in how the budgets of uncertainty are imposed. In their approach, the budget is utilized to constrain the sum of unscaled deviations. As a consequence, gradients for the objective function can be obtained by solving a simple knapsack problem.

Focusing on robust optimization results in the newsvendor problem, Perakis and Roels (2008) [17] present a framework that unifies several previously known results and derive robust order quantities for the cases of known support, known support and mean, known mean and median, known support and unimodality. They also characterize the value of knowledge of the shape of the distribution. Their approach does not follow that found in reference [7] and others who use maximin of a profit function, but instead is inspired by the work of Savage (1951) [20], using minimax of regret as the criterion to select a distribution from a convex set of distributions $\mathcal{D}$. Two interesting observations of their work are 1) The most robust distributions are in general also entropy maximizing and 2) The ability to include qualitative information (symmetry, skewness, etc.) about the shape of the distribution.

In a similar line of work, Natarajan et al (2009) [16] analyze the robust newsvendor problem and attempt to extend known results to include information on asymmetry such as skewness and kurtosis. While it is typically hard to find closed-form expressions on the performance of such models against the optimal solution, the authors present a framework that is able to model asymmetric demand with a piecewise linear objective by deriving uncertainty sets through knowledge of mean, variance, and semivariance, and are able to use the knowledge of asymmetry to reduce the degree of conservatism in the answer while attaining explicit lower bounds on the expected profit.

Bienstock and Özbay (2008) [9] study a generalization of the Bertsimas and Thiele paper. Their model allows for non-stationary costs over time and admits efficient algorithms based on Bender's decomposition that compute optimal solutions. Additionally, for very large problems, Bienstock and Özbay demonstrate near optimality of an approximation scheme. In addition to generalizing the Bertsimas-Thiele model found in reference [8], the authors also study another type of uncertainty set which they term the *bursty demand* setting. In this setting, the authors define intervals over which there is at most one exceptional period which experiences a peak demand much larger than the mean.

Mamani et al (2016) [15] also study a robust inventory management which generalizes certain results of Bertsimas and Thiele [8]. Notably, the Mamani et al. model incorporates correlated demands over multiple periods while the model in reference [8] required i.i.d.

demands. Additionally, the Mamani et al. model does not require the uncertainty sets to be symmetric. However, the model focuses only on a single product at a single location subject to holding, purchasing, and shortage costs. Remarkably, the authors were able to derive closed form expressions for determining the optimal ordering quantity in each period using only the means and covariance matrix for the demand in each period.

One of the more recent ideas that have been applied to robust optimization is that of uncertainty sets inspired by the Central Limit Theorem. To our knowledge, Rikun's doctoral thesis (2011) [18] was the first to do so, creating Central Limit Theorem inspired bounds on the aggregate deviation in terms of standard deviations of a demand process over time and applying these uncertainty sets to analyze the performance of queueing systems. Rikun extends these formulations in a later work, Bertsimas et al. (2011) [5], by considering the implications of the Law of the Iterated Logarithm in the context of performance analysis of a queueing network. Here, the Law of the Iterated Logarithm applies to the underlying stochastic queueing model while generating linear constraints in the robust optimization model i.e. $\sum_{1 \le i \le k} U_i \le \lambda^{-1} k + \Gamma \sqrt{k \ln \ln k}$, for all $k \ge 1$ where U_i is a stochastic primitive such as the sequence of interarrival times and λ is the arrival rate of the primitive. Using this model, Bertsimas et al. are able to generate closed form bounds on performance measures in several different classes of queueing networks. Bandi and Bertsimas (2012) [1] further generalize this approach by considering other probability limit laws that govern the underlying stochastic queueing process and form linear constraints in the robust optimization model. Bandi and Bertsimas also consider uncertainty sets inspired by the Central Limit Theorem, Nolan's generalized Central Limit Theorem, and Shannon's typical sets from information theory. These uncertainty sets are able to handle a wide variety of settings, incorporating correlated random variables and heavy tailed processes, and provide explicit bounds on the performance of the underlying queueing network.

Recently, Jackson and Muckstadt (2016) [14] solve a robust multi-period, two echelon stock allocation problem where they relax the typically assumed Clark-Scarf balance assumption. The balance assumption requires that there is no desire to reallocate stock from one seller to another during any time period. However, under highly variable demand processes, imbalances of inventory may occur, leading to cases where a few sellers have high levels of inventory while others may incur backorders. In these cases, rebalancing may be desirable. The uncertainty set used in this paper is termed the "multi-period risk pooling uncertainty set" which restricts the deviation in demand among any collection of products across time periods to be less than the product of an upper bound δ and the square root of the product of the cardinality of the set of products and the total number of time periods. The problem is first formulated with this uncertainty set as a linear program with an exponential number of constraints. The problem is later reformulated iteratively as a master problem which is a simple linear program and a mixed integer program sub-problem.

Notably, Jackson and Muckstadt study the inventory allocation problem in the recourse formulation which differs from the previously mentioned papers. The recourse formulation with robust optimization allows their model to capture a risk pooling effect - because no transshipments can occur, there is incentive to hold back inventory at a central location in order to have inventory available for sellers in future periods. Their approach is inspired by Bredström et al (2013) [10] and partitions the variables into a master linear problem with business decisions made without knowledge of demand and a recourse problem which is a bilinear program. However, the formulation in reference [14] can be further simplified into a master linear problem and a mixed integer program sub-problem. This is contrasted with the approach taken by Bertsimas and Thiele as well as others where the cost after demand is realized is considered together with the inventory allocation decision.

This paper is largely inspired by reference [14] and attempts to address one possible criticism of their model. The uncertainty set used in their work is inspired by the typical square root laws seen in central limit theorems and has the following form:

$$U(\delta) = \left\{ \boldsymbol{\varepsilon} : \begin{array}{c} \varepsilon_{it} \leq \delta, \forall (i,t) \\ \sum_{i \in I} \sum_{t=1}^{t'} \varepsilon_{it} \leq \sqrt{|I| t'} \delta, \ \forall \ I \subseteq \mathcal{N}, t' = 1, \ldots, T \end{array} \right\} \tag{1.1}$$

where ε_{it} describes the deviation of the demand realization for product i in period t from its mean and $\mathcal{N}$ is the set of sellers. Summarizing, the demand for item i in period t, d_{it}, is modeled as

$$d_{it} = \mu_{it} + \sum_{j \in \mathcal{N}} c_{ijt} \varepsilon_{jt}.$$

where μ_{it} is the average demand for product i in period t and c_{ijt} is the (i, j) entry of the Cholesky factorization, C, of the covariance matrix Σ_t. That is, $\Sigma_t = C_t C_t^T$.

Examining the last constraint of the uncertainty set described by 1.1, we see that it is motivated by the Central Limit Theorem. Specifically, as independent items are aggregated, while the mean of the sum is the sum of the means, the standard deviation of the sum grows as the square root of the sum of standard deviations. However, as the number of products increases, the probability that the random demand seen in each period lies within the uncertainty set goes to 0. That is, as the number of constraints increases, the probability that at least one constraint is violated increases. For instance, there is a constraint on how far any individual demand can deviate from the mean. Then if the demand for item i in periods 1 through t has cumulative distribution function F_t, the probability that the sum of demands does not violate the uncertainty set is $F^{-1}(\sqrt{t}\delta)$. Then using the independence of demand across periods, the probability of satisfying the constraint for all t is $\prod_{t=1}^{T} F^{-1}(\sqrt{t}\delta)$. Thus, as the number of items and number of time periods grows, the probability that demand satisfies all constraints goes to zero. Therefore, we propose a new uncertainty set that addresses this modeling issue. We then use this uncertainty set to solve the single-period multi-item newsvendor problem.

The rest of the paper is organized as follows. We begin with a brief motivation for the problem setting that we analyze, followed with a discussion of the uncertainty set and comparisons to existing uncertainty sets found in the literature. We begin the analysis of the problem under our chosen uncertainty set for the single item case. After deriving results and closed form ordering quantities for the single item case, we analyze the multi-item case. We begin our multi-item analysis with two identical items and then extend the analysis to a two non-identical item case. We then analyze the general multi-item case and derive a 2-approximation for the optimal cost to the seller through Lagrangian relaxation. We demonstrate the asymptotic optimality of our approach and conclude with numerical experiments.

2 Motivation

Planning inventory levels for many items that have highly varying and low demand rates is a significant challenge. For example, this challenge is faced by inventory managers within the United States Air Force. Currently, models used by these managers to determine inventory levels require explicit knowledge of the stochastic processes governing the failure patterns of expensive repairable items.

However, many changes have occurred over the preceding decades to the logistics system employed by the United States Air Force to maintain its fleets of aircraft. Due to cost considerations, the United States Air Force has reduced the amount of active aircraft as well as the number of flying missions. Additionally, development costs for newer aircraft are increasing, and the technological abilities of each unit of aircraft have increased greatly. For example, the F-22 fighter jet which was first introduced in 2005 was produced at a cost of $150 million per airframe with a total cost (including development) of $66.7 billion. As a result, the total production was reduced to only 195 units. Similarly, the F-35 fighter jet has been allocated a program cost of $1.508 trillion with only 231 units produced as of March 2017. This is contrasted with earlier fighter jets such as the F-15 (1,198 units built at a unit cost of \$29-30 million) and the F-16 fighter jet (4,573 units built at a unit cost of \$15-19 million). Trends like these have lead to very high development costs and consequently, relatively low numbers of aircraft stationed at each base. Furthermore, we note the following facts:

(i) Manpower costs have risen substantially.

(ii) Newer weapons systems are technologically advanced and this trend will continue, necessitating additional parts on each aircraft.

(iii) The number of aircraft of a particular model design series that are now procured is not large. These aircraft are operated at many locations throughout the world, and sometimes deployed in small numbers.

(iv) Technology advances are accompanied by a requirement for highly skilled technicians and complex repair equipment.

(v) Repairs of certain failed items are conducted by contractors.

(vi) Flying activity has been curtailed due to budget limitations.

The result of these trends and facts is that presently there are fewer aircraft employed by the United States Air Force than before, each with more parts and system complexity. The facts listed above lead us to conclude that these trends will persist into the future. The result on the logistics system is therefore twofold: 1) There amount of data being gathered on real-world part performance and failure rates is greatly reduced, and 2) The additional system complexity can lead to a highly variable demand process and noisier data. Furthermore, in recent years, the US Air Force has been deploying aircraft for short periods of time to many locations in less than squadron numbers. These deployed aircraft have highly variably day-to-day flying activities which impact the number of failures of items installed on these aircraft. Additionally, estimates of failure rates may be determined using data collected from environments that differ greatly from the actual operating environments.

Therefore, as a result of these challenges, we believe that the traditional stochastic model may not capture the true dynamics of this system. As a result, we propose studying the problem facing these inventory managers through a robust planning model which captures a demand process with more uncertainty than found when using a Markovian stochastic model, as is often done in practice. In the Air Force setting, repair of failed items normally occurs at a centralized location and resupply of the base stock for these failed items comes from this centralized facility. Hence, the problem of setting stocking levels for items can be represented as a newsvendor-like problem, as we will now demonstrate.

The problem faced by inventory managers for the US Air Force is to select a set of stock levels for critical repairable aircraft parts subject to a budget constraint. By selecting appropriate stock levels, $\boldsymbol{s}$, these managers seek to minimize the number of backorders, each of which can result in a non-operational aircraft. That is, they face the following problem:

$$\begin{aligned} \min_{\boldsymbol{s}} & \sum_{i=1}^{n} \mathbb{E}\left[(d_i - s_i)^+\right] \\ \text{subject to} & \sum_{i=1}^{n} s_i c_i \leq B, \end{aligned}$$

where d_i represents the demand for item i, B represents the available budget, and c_i represents the per-unit cost for item i. One method that is often employed to solve these problems is to allocate inventory through a marginal analysis approach. However, observe that by dualizing the budgets constraint with a Lagrange multiplier θ, we obtain the following problem:

$$\min_{\boldsymbol{s}} \sum_{i=1}^{n} \mathbb{E}\left[(d_i - s_i)^+\right] + \sum_{i=1}^{n} \left[\theta c_i s_i\right] - \theta B,$$

where the choice of θ results in an imputed holding cost rate of h_i, which is approximately equal to θc_i. Thus, this problem can also be viewed as an unconstrained newsvendor-like problem. We now describe our problem setting in more detail.

3 Model

3.1 The Uncertainty Set

As mentioned before, one of the concerns related to the uncertainty set pertaining to the demand process used in reference [14] is that as the number of items considered by the seller grows, the probability that the deviations described by ε lie within the described uncertainty set go to 0. Therefore, we consider an alternative uncertainty set for the demand process, $U(\boldsymbol{\delta}, \boldsymbol{\delta_Z})$, to be defined below, where $\boldsymbol{\delta}$ is a vector containing $\delta_i^U, \delta_i^L \; \forall \; i$ and $\boldsymbol{\delta_Z}$ is a vector containing δ_Z^+, δ_Z^-. These parameters will be described in greater detail shortly. We will continue to use bold font to denote vectors throughout the remainder of this paper. The motivation for this uncertainty set is similar to that of Bertsimas and Thiele where the total deviations in demand in any period are bounded by individual upper and lower bounds as well as a joint constraint that can be thought of as a budget.

Suppose that the demand for item i is of the form

$$d_i = \mu_i + \sigma_i \varepsilon_i, \; \boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \boldsymbol{\delta_Z})$$

where ε_i is chosen by an adversary whom we will refer to as "nature." We restrict the demand realizations for each item to lie within a certain interval. That is, we impose upper and lower bounds on the normalized deviation from the mean, ε_i. From the above expression for d_i, we see that the scaled deviation from the mean is the quantity $\sigma_i \varepsilon_i$. Therefore, we will utilize constraints of the form:

$$-\delta_i^L \leq \varepsilon_i \leq \delta_i^U \; \forall \; i.$$

Additionally, drawing upon the Central Limit Theorem, we bound the aggregate deviation through joint constraints, or "budgets," on all items. The previously discussed literature has typically proposed uncertainty sets which utilize a single constraint bound the aggregate deviations from the mean, treating deviations above and below the mean in the same manner.

However, we bound the deviations from the mean separately in terms of demands larger than the mean and demands smaller than the mean. This approach permits us to have increased control of the demand process by tuning both budgets of uncertainty. In our motivation for this problem, mean demand rates are low, preventing demand from falling significantly below the mean as demand must be non-negative. Additionally, backorder costs are substantially higher than holding costs. As a result, we believe it to be more useful to treat large and small demands separately.

Another justification for our type of uncertainty set is that we are looking at a setting in which there are a low number of flying missions. Therefore, while we could employ the Central Limit Theorem to bound the total deviation through constraints of the form

$$\underline{\Gamma} \leq \sum_{i=1}^{n} \sigma_i \varepsilon_i \leq \overline{\Gamma},$$

where n is the number of items and ε_i is a normally distributed variable, this may be overly optimistic as the number of flying missions may be too low for the aggregate demand distribution to be approximately normal. Our more conservative approach relies on the observation that for total demand D,

$$\sum_{i=1}^{n} \left[\mu_i - \sigma_i \varepsilon_i^-\right] \leq D \leq \sum_{i=1}^{n} \left[\mu_i + \sigma_i \varepsilon_i^+\right].$$

where ϵ_i^+ is the maximum of a standard normal random variable and 0, ϵ_i^- is the minimum of a standard normal random variable and 0. This then allows the possibility of more variable demands than the Central Limit Theorem would suggest. We then consider upper bounds on $\sum_{i=1}^{n} \sigma_i \varepsilon_i^-$ and $\sum_{i=1}^{n} \sigma_i \varepsilon_i^+$. For these bounds, we resort to the Central Limit Theorem, but applied to ε_i^+ and ε_i^-.

We now introduce $\tilde{Z}^+ = \sum_{i=1}^{n} \sigma_i \epsilon_i^+$ and $\tilde{Z}^- = \sum_{i=1}^{n} \sigma_i \epsilon_i^-$ with corresponding scalars δ_Z^+, δ_Z^-, respectively. As stated above, ϵ_i^+ is the maximum of a standard normal random variable and 0, ϵ_i^- is the minimum of a standard normal random variable and 0. Note that this differs from our previously defined ε_i which represents the deviation from the mean. We add the following constraints:

$$
\begin{aligned}
\sum_{i=1}^{n} \varepsilon_i^+ \sigma_i &\leq \mathbb{E}\left[\tilde{Z}^+\right] + \delta_Z^+ \sqrt{\mathrm{Var}\left(\tilde{Z}^+\right)}, \\
\sum_{i=1}^{n} \varepsilon_i^- \sigma_i &\leq \left|\mathbb{E}\left[\tilde{Z}^-\right]\right| + \delta_Z^- \sqrt{\mathrm{Var}\left(\tilde{Z}^-\right)}, \\
\varepsilon_i &= \varepsilon_i^+ - \varepsilon_i^- \ \forall\ i, \\
\varepsilon_i^- &\geq 0 \ \forall\ i, \\
\varepsilon_i^+ &\geq 0 \ \forall\ i.
\end{aligned}
$$

For ease of exposition, we will refer to the right-hand side of the first two constraints as C^+ and C^-, respectively. Next, we define some terms. Note that for any realization of ε_i, the resulting items can be partitioned into those with demands larger than their means and those with demands less than or equal to their means. Define

$$
\begin{aligned}
S^+ &= \{i : d_i > \mu_i\} \text{ and} \\
S^- &= \{i : d_i \leq \mu_i\}.
\end{aligned}
$$

Then S^- is the set of items with demands less than or equal to their means and S^+ is the set of items with demands greater than their means. Turning back to the constraints above, the first constraint bounds the total amount by which the demand realizations for items in S^+ can exceed the sum of their means. Likewise, the second constraint bounds the total amount by which the demand realizations for items in S^- can fall short of the sum of their means. The third constraint simply ties the two deviation variables ε_i^+ and ε_i^- for an item i into a total amount of deviation ε_i. Putting these constraints together, we obtain the following uncertainty set:

$$U(\boldsymbol{\delta}, \boldsymbol{\delta_Z}) = \left\{ \boldsymbol{\varepsilon} : \begin{array}{c} \sum_{i=1}^{n} \varepsilon_i^+ \sigma_i \leq \mathbb{E}\left[\tilde{Z}^+\right] + \delta_Z^+ \sqrt{\mathrm{Var}\left(\tilde{Z}^+\right)} \\ \sum_{i=1}^{n} \varepsilon_i^- \sigma_i \leq \left|\mathbb{E}\left[\tilde{Z}^-\right]\right| + \delta_Z^- \sqrt{\mathrm{Var}\left(\tilde{Z}^-\right)} \\ 0 \leq \varepsilon_i^+ \leq \delta_i^U \ \forall\, i \\ 0 \leq \varepsilon_i^- \leq \delta_i^L \ \forall\, i \\ \varepsilon_i = \varepsilon_i^+ - \varepsilon_i^- \ \forall\, i \\ \varepsilon_i^+, \varepsilon_i^- \geq 0 \ \forall\, i \end{array} \right\}. \tag{3.1}$$

Throughout the majority of the analysis, to ease the exposition, we will assume that $\delta_Z^+ = \delta_Z^-$ and use a single parameter δ_Z. Hence, while this assumption holds, $C^+ = C^-$ and we will refer to a single budget using C. We will subsequently relax this assumption in Section 4.3. This results in the following uncertainty set:

$$U(\boldsymbol{\delta}, \delta_Z) = \left\{ \boldsymbol{\varepsilon} : \begin{array}{c} \sum_{i=1}^{n} \varepsilon_i^+ \sigma_i \leq \mathbb{E}\left[\tilde{Z}^+\right] + \delta_Z \sqrt{\mathrm{Var}\left(\tilde{Z}^+\right)} \\ \sum_{i=1}^{n} \varepsilon_i^- \sigma_i \leq \left|\mathbb{E}\left[\tilde{Z}^-\right]\right| + \delta_Z \sqrt{\mathrm{Var}\left(\tilde{Z}^-\right)} \\ 0 \leq \varepsilon_i^+ \leq \delta_i^U \ \forall\, i \\ 0 \leq \varepsilon_i^- \leq \delta_i^L \ \forall\, i \\ \varepsilon_i = \varepsilon_i^+ - \varepsilon_i^- \ \forall\, i \\ \varepsilon_i^+, \varepsilon_i^- \geq 0 \ \forall\, i \end{array} \right\}. \tag{3.2}$$

As before, $\boldsymbol{\delta}$ is a vector containing $\delta_i^U, \delta_i^L \ \forall\, i$ and δ_Z is a parameter that is chosen by the decision maker to reflect nature's ability to impact demand uncertainty.

Note that the right-hand side of the joint constraints can be thought of as a "risk budget" or perhaps an appetite for risk. This has two components - one is a non-scalable portion $\mathbb{E}[\tilde{Z}^+]$ and the other is scaled by the parameter δ_Z which we will assume to be non-negative. We will discuss this observation further as we develop our analysis.

While $\boldsymbol{\delta}$ is often thought of as a protection level (e.g. standard deviations above the mean) or tolerance for risk in the classic supply chain management literature, it is important to note that this is not necessarily the case in the robust optimization problem. While the inventory level for item i may increase as the values of δ_i^U and δ_i^L vary, δ_i^U and δ_i^L can also be thought of as the degree of uncertainty or variability associated with the demand process caused by nature. That is, if an item's demand parameters cannot be estimated accurately, δ may be quite large due to this uncertainty rather than a protection level.

We begin our analysis by determining the first two moments of ϵ_i^+. To do so, we integrate by parts:

$$\mathbb{E}[\epsilon_i^+] = \int_0^\infty x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \left[-e^{-x^2/2} \right|_0^\infty - 0 = \frac{1}{\sqrt{2\pi}} \quad \text{and}$$
$$\mathbb{E}[(\epsilon_i^+)^2] = \int_0^\infty x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \left[-\frac{1}{\sqrt{2\pi}} x e^{-x^2/2} \right|_0^\infty + \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{2}.$$

Since all ϵ_i are i.i.d., it follows that $\mathrm{Var}(\epsilon_i^+) = \frac{1}{2}\left(1 - \frac{1}{\pi}\right)$ and

$$\mathbb{E}\left[\tilde{Z}^+\right] = \sqrt{\frac{1}{2\pi}} \sum_{i=1}^n \sigma_i \text{ and}$$
$$\mathrm{Var}\left(\tilde{Z}^+\right) = \frac{1}{2}\left(1 - \frac{1}{\pi}\right) \sum_{i=1}^n \sigma_i^2.$$

By symmetry, $\left|\mathbb{E}\left(\tilde{Z}^-\right)\right| = \sqrt{\frac{1}{2\pi}} \sum_{i=1}^n \sigma_i$ and $\mathrm{Var}\left(\tilde{Z}^-\right) = \frac{1}{2}\left(1 - \frac{1}{\pi}\right) \sum_{i=1}^n \sigma_i^2$.

It should be noted that when compared to $\tilde{Z} = \sum_{i=1}^n \epsilon_i \sigma_i$ where ϵ_i are i.i.d. standard normals,

$$\mathbb{E}[\tilde{Z}] = 0 < \mathbb{E}[\tilde{Z}^+]$$
$$\mathrm{Var}(\tilde{Z}) = \sum_{i=1}^n \sigma_i^2 > \mathrm{Var}(\tilde{Z}^+).$$

As stated previously, we have a non-scalable positive component in the risk budget corresponding to our more conservative approach. If we had chosen to bound the deviations in this other manner, we would have only a single scalable component in the risk budget.

Now that we have characterized the uncertainty set, we proceed to the problem setting.

3.2 Problem Setting

In the problem that we consider, a seller seeks to choose the appropriate inventory level in order to minimize the worst case cost in the next period where nature chooses demand from the uncertainty set $U(\boldsymbol{\delta}, \delta_Z)$. Specifically, we assume that orders for inventory are satisfied instantaneously at the beginning of each period, and without loss of generality, at no cost to the seller. Additionally, the seller has end of period backorder costs b_i and holding costs h_i that may vary by item. We assume that $\min_i b_i > \max_i h_i > 0$ and, as stated above, that demand is of the form $\boldsymbol{d} = \boldsymbol{\mu} + \boldsymbol{\sigma\varepsilon}, \boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \delta_Z)$.

The seller's problem then can be expressed as

$$\begin{aligned} \min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} \quad & \sum_{i=1}^{n} b_i(\mu_i + \varepsilon_i\sigma_i - s_i)^+ + h_i(s_i - \mu_i - \varepsilon_i\sigma_i)^+ \\ \text{subject to} \quad & \boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \delta_Z). \end{aligned} \tag{3.3}$$

Observe that at most one of the two terms indexed by i can be positive. As a result, the seller's problem can be rewritten as

$$\begin{aligned} \min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} \quad & \sum_{i=1}^{n} \max\left\{b_i(\mu_i + \varepsilon_i\sigma_i - s_i), h_i(s_i - \mu_i - \varepsilon_i\sigma_i)\right\} \\ \text{subject to} \quad & \boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \delta_Z). \end{aligned} \tag{3.4}$$

It is then easy to see that nature's maximization problem is convex with respect to $\boldsymbol{\varepsilon}$ since it is a maximum of two linear functions which correspond to the backorder and holding costs. For each item, nature observes the seller's chosen stock levels and seeks to perturb the demand from the expected value, μ, to some new value (higher or lower) that maximizes the cost to the seller. The convexity of nature's maximization problem indicates that $\boldsymbol{\varepsilon}$ must lie on the boundary of $U(\boldsymbol{\delta}, \delta_Z)$. This can be easily seen as any point where the gradient is zero must be a global minimum. There is another easy way to see this is the case. Due to convexity, any interior point x can be expressed as a weighted sum of the extreme points.

That is, $x = \sum_{i \in E} w_i x_i^*$ where E is the set of extreme points and $\sum_i w_i = 1$. Furthermore, because the maximization is convex, we can utilize Jensen's inequality to show that the value at any interior point x is smaller than the weighted sum of the function evaluated at the extreme points, indicating that unless the function is constant, at least one of the extreme points has larger value. For a more detailed proof, see Rockafellar (1970) [19].

We now demonstrate that the outer minimization is convex with respect to $\boldsymbol{s}$.

Lemma 3.1 *The seller's minimization problem given in (3.4) is convex with respect to* $\boldsymbol{s}$*.*

Proof. Because $d_i = \mu_i + \varepsilon_i \sigma_i$, problem (3.4) can be written as:

$$\begin{aligned} \min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} \quad & \sum_{i=1}^{n} \max\left\{b_i(d_i - s_i), h_i(s_i - d_i)\right\} \\ \text{subject to} \quad & \boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \delta_Z). \end{aligned} \tag{3.5}$$

Consider the two vectors of stock levels $\hat{\boldsymbol{s}}$ and $\overline{\boldsymbol{s}}$, and denote the cost expression by

$$f(\boldsymbol{s}) = \max_{\boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \delta_Z)} \sum_{i=1}^{n} \max\left\{b_i(d_i - s_i), h_i(s_i - d_i)\right\}.$$

Now define a variable $\lambda \in [0, 1]$. Then $\lambda \hat{\boldsymbol{s}} + (1-\lambda)\overline{\boldsymbol{s}}$ is a convex combination of $\hat{\boldsymbol{s}}$ and $\overline{\boldsymbol{s}}$, and f evaluated at this point is:

$$f(\lambda \hat{\boldsymbol{s}} + (1-\lambda)\overline{\boldsymbol{s}}) = \max_{\boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \delta_Z)} \sum_{i=1}^{n} \max\left\{b_i(d_i - \lambda \hat{s}_i - (1-\lambda)\overline{s}_i), h_i(\lambda \hat{s}_i + (1-\lambda)\overline{s}_i - d_i)\right\}.$$

Observe that

$$\begin{aligned} b(d_i - \lambda \hat{s}_i - (1-\lambda)\overline{s}_i) &= b_i(d_i - \lambda \hat{s}_i - (1-\lambda)\overline{s}_i) + \lambda b_i d_i - \lambda b_i d_i \\ &= \lambda b_i(d_i - \hat{s}_i) - (1-\lambda) b_i \overline{s}_i + (1-\lambda) b_i d_i \\ &= \lambda b_i(d_i - \hat{s}_i) + (1-\lambda) b_i(d_i - \overline{s}_i). \end{aligned}$$

Similarly, $h_i(\lambda \hat{s}_i + (1-\lambda)\overline{s}_i - d_i) = \lambda h_i(\hat{s}_i - d_i) + (1-\lambda)h_i(\overline{s}_i - d_i)$. Thus, substituting these into our previous expression:

$$f(\lambda \hat{\boldsymbol{s}} + (1-\lambda)\overline{\boldsymbol{s}}) = \max_{\boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \delta_Z)} \sum_{i=1}^{n} \max \left\{ \begin{matrix} \lambda b_i(d_i - \hat{s}_i) + (1-\lambda) b_i(d_i - \overline{s}_i), \\ \lambda h_i(\hat{s}_i - d_i) + (1-\lambda) h_i(\overline{s}_i - d_i) \end{matrix} \right\}.$$

Recall that for real a_1, a_2, b_1, b_2, $\max\{a_1 + a_2, b_1 + b_2\} \leq \max\{a_1, b_1\} + \max\{a_2, b_2\}$. Using this observation,

$$f(\lambda \hat{\boldsymbol{s}} + (1-\lambda)\overline{\boldsymbol{s}}) \leq \max_{\boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \delta_Z)} \sum_{i=1}^{n} \left\{ \begin{matrix} \lambda \max\{b_i(d_i - \hat{s}_i), h_i(\hat{s}_i - d_i)\} \\ +(1-\lambda) \max\{b_i(d_i - \overline{s}_i), h_i(\overline{s}_i - d_i)\} \end{matrix} \right\}.$$

Then, splitting the maximization with respect to $\boldsymbol{\varepsilon}$,

$$\begin{aligned} f(\lambda \hat{\boldsymbol{s}} + (1-\lambda)\overline{\boldsymbol{s}}) &\leq \begin{matrix} \lambda \max_{\boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \delta_Z)} \left\{ \sum_{i=1}^{n} \max\{b_i(d_i - \hat{s}_i), h_i(\hat{s}_i - d_i)\} \right\} \\ +(1-\lambda) \max_{\boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \delta_Z)} \left\{ \sum_{i=1}^{n} \max\{b_i(d_i - \overline{s}_i), h_i(\overline{s}_i - d_i)\} \right\} \end{matrix} \\ &= \lambda f(\hat{\boldsymbol{s}}) + (1-\lambda) f(\overline{\boldsymbol{s}}). \end{aligned}$$

Thus $f(\cdot)$ is convex with respect to $\boldsymbol{s}$. ■

4 Analysis

In the previous section, we discussed the intuition behind the chosen uncertainty set and compared the uncertainty set to those found in the literature. The results from the previous section also indicate that there is a unique optimal cost for any problem setting. In this section, we build upon the previous analysis to find closed form expressions for the optimal stock levels.

We begin our analysis with a gentle introduction to the flexibility permitted by our uncertainty set. We begin by analyzing the single item case under the assumptions that

$\delta_Z^+ = \delta_Z^-$ and $\delta^U = \delta^L$ to develop techniques and intuition. We will then use the same techniques to analyze the two identical items when both items have identical cost and demand parameters. We then relax the assumption that $\delta^U = \delta^L$. Next, we continue with the analysis of a two non-identical item case where demand parameters may vary by item, but cost parameters are identical, still under the assumption that $\delta_Z^+ = \delta_Z^-$. Finally, we relax the assumption on cost parameters and allow item dependent cost parameters. We end the analysis section by demonstrating that the optimal solution in the general multi-item case cannot be found easily. We apply Lagrangian relaxation to the general problem to derive an asymptotically optimal policy with a 2-approximation performance guarantee that performs well in practice.

4.1 Single-Item

First, we consider the single-item, single-period robust newsvendor problem where it has been assumed that $\delta_Z^+ = \delta_Z^-$ and $\delta^U = \delta^L$. Hence, we will utilize the parameters δ_Z and δ to represent these quantities, respectively. In this case, Problem 3.4 is:

$$\begin{aligned} \min_{s} \max_{\varepsilon} \quad & \max\{b(d-s), h(s-d)\} \\ \text{subject to} \quad & \varepsilon \in U(\delta, \delta_Z). \end{aligned}$$

But now note that for a single item, $U(\delta, \delta_Z)$ has special structure. Modifying the uncertainty set presented in (3.2),

$$U(\delta, \delta_Z) = \left\{ \varepsilon : \begin{array}{c} \varepsilon^+ \sigma \le \mathbb{E}\left[\tilde{Z}^+\right] + \delta_Z \sqrt{\mathrm{Var}\left(\tilde{Z}^+\right)} \\ \varepsilon^- \sigma \le \left|\mathbb{E}\left[\tilde{Z}^-\right]\right| + \delta_Z \sqrt{\mathrm{Var}\left(\tilde{Z}^-\right)} \\ 0 \le \varepsilon^+ \le \delta \\ 0 \le \varepsilon^- \le \delta \\ \varepsilon = \varepsilon^+ - \varepsilon^- \\ \varepsilon^+, \varepsilon^- \ge 0 \end{array} \right\}.$$

In this scenario, observe that only one constraint can be active. That is, $\varepsilon \in U(\delta, \delta_Z)$ can be expressed as $|\varepsilon| \le c$ where $c = \min \left\{ \delta, \frac{\mathbb{E}[\tilde{Z}^+] + \delta_Z \sqrt{\mathrm{Var}(\tilde{Z}^+)}}{\sigma} \right\}$. But recall that $\mathbb{E}\left[\tilde{Z}^+\right] = \frac{1}{\sqrt{2\pi}}\sigma$ and that $\mathrm{Var}\left(\tilde{Z}^+\right) = \frac{1}{2}(1 - \frac{1}{\pi})\sigma^2$. Therefore we can simplify the expression to $c = \min \left\{ \delta, \frac{1}{\sqrt{2\pi}} \left(1 + \delta_Z \sqrt{\pi - 1}\right) \right\}$.

In the previous section, we noted that ε must lie on the boundary, and therefore $\varepsilon = \pm c$. Having characterized how nature will react for any given stock level s, we can now find the optimal choice of stock level.

Proposition 4.1 *The optimal choice of s is $s^* = \mu + \frac{b-h}{b+h} c\sigma$ with optimal cost of $\frac{2bh}{b+h} c\sigma$.*

Proof. First, observe that the cost to the seller is identical under $\varepsilon = c$ and $\varepsilon = -c$ when $s = s^*$. That is,

$$\begin{aligned} h(s^* - (\mu - c\sigma)) &= h\left((\mu + \frac{b-h}{b+h} c\sigma) - \mu + c\sigma \right) \\ &= h\left(\left(\frac{b-h}{b+h} + 1 \right) c\sigma \right) = \frac{2bh}{b+h} c\sigma \end{aligned}$$

and

$$\begin{aligned} b((\mu + c\sigma) - s^*) &= b\left(\mu + c\sigma - (\mu + \frac{b-h}{b+h} c\sigma) \right) \\ &= b\left(\left(1 - \frac{b-h}{b+h} \right) c\sigma \right) = \frac{2bh}{b+h} c\sigma. \end{aligned}$$

Because the optimal solution for nature must lie on the boundary, the only possible optimal actions for nature are $\varepsilon = \pm c$. Thus, the cost function can be rewritten $f(s) = \max\{b(\mu + c\sigma - s), h(s - \mu + c\sigma)\}$. We depict the cost function graphically below as a function of ending stock level.

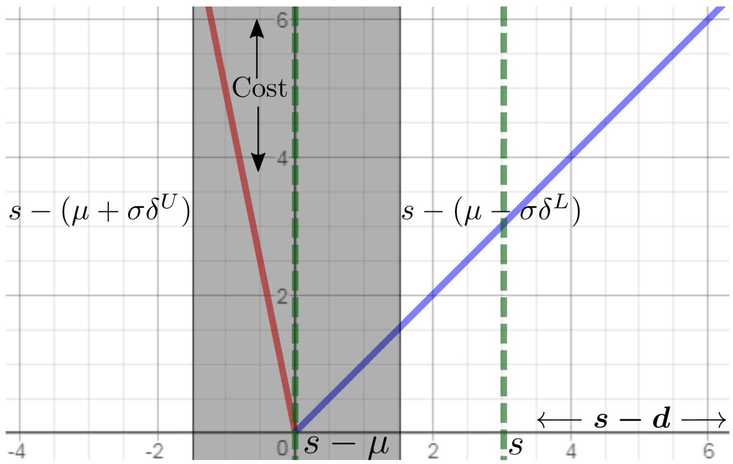

Figure 1: Depiction of stock level vs cost function for $\mu = 3, \sigma = 1, C^+ = C^- = 1.5, b = 5, h = 1, \delta^U = \delta^L = 2$ when $s = \mu$ (shown in green). Backorder cost $b(\mu + c - s)$ in red, holding cost $h(s + \mu - c)$ in blue.

Recall that because of the convexity of nature's maximization problem, the optimal ε must lie on the boundary. Therefore, $\varepsilon = \pm c$ with the determination made by nature to maximize the cost. This corresponds to the maximum of the two linear functions above and is depicted by the solid lines in the graph above. Note that in the above figure, the seller has chosen to stock the mean demand. Therefore, if the demand is exactly equal to the mean, the seller will have a ending stock level of 0, incurring no additional costs. However, because nature is able to choose a demand within the uncertainty set, nature can freely choose any value in the shaded region to be the realized end of period stock level.

Observe that the backorder cost is decreasing in s while the holding cost is increasing in s. Therefore, in order to minimize the cost to the seller, the seller should choose the stock level that makes nature indifferent between the backorder and holding costs. Graphically, this is represented below:

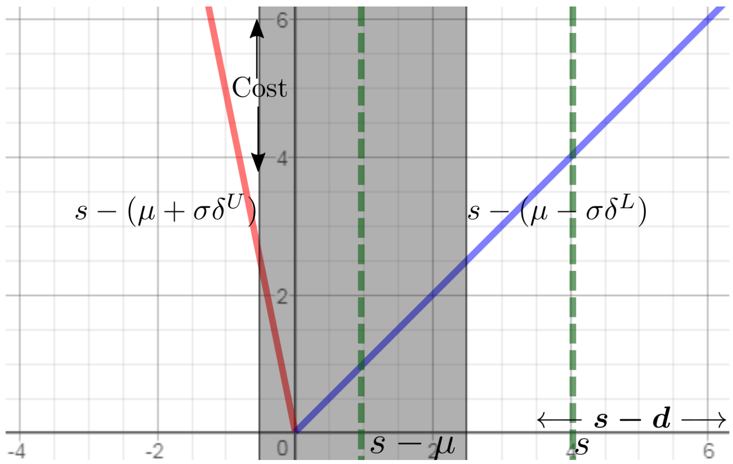

Figure 2: Depiction of stock level vs cost function for $\mu = 3, \sigma = 1, C^+ = C^- = 1.5, b = 5, h = 1, \delta = 2$ when $s = s^*$ (shown in green). Backorder cost $b(\mu + c - s)$ in red, holding cost $h(s + \mu - c)$ in blue.

In the above figure, the seller has chosen to stock the optimal stock level. Note that because this is the optimal stock level, the two extremes for nature ($\varepsilon = \delta, \varepsilon = -\delta$) result in equal costs to the seller.

Stated mathematically, to minimize $f(s)$, s must satisfy $b(\mu + c\sigma - s) = h(s - \mu + c\sigma)$, which results in $s = s^*$. ■

Corollary 4.1 *If $s < s^*, \varepsilon = c$ and if $s > s^*, \varepsilon = -c$. That is, if the stock level is lower than optimal, nature seeks to maximize backorders. If the stock level is higher than optimal, nature seeks to maximize holding costs.*

Recall from the above discussion that $c = \min\left\{\delta, \frac{1}{\sqrt{2\pi}}\left(1 + \delta_Z\sqrt{\pi - 1}\right)\right\}$ which leads to the following corollary.

Corollary 4.2 *If $\delta \geq \frac{1}{\sqrt{2\pi}}\left(1 + \delta_Z\sqrt{\pi - 1}\right)$, then*

$$s^* = \mu + \frac{b-h}{b+h}\frac{\sigma}{\sqrt{2\pi}}\left(1 + \delta_Z\sqrt{\pi - 1}\right).$$

Otherwise,

$$s^* = \mu + \frac{b-h}{b+h}\sigma\delta.$$

Observe that for large δ, there is a non-scalable portion of the safety stock that is due to the conservatism of the approach. That is, due to the conservatism of the model, even if $\delta_Z = 0$, there will still be a positive amount of safety stock.

This leads us to a discussion of the roles of δ and δ_Z. As mentioned before, δ_Z can be thought of as a risk-budget that is set through a managerial decision. However, δ in this case can be thought of in two ways. One is that of the traditional stochastic model in which δ can be thought of as a tolerance for risk or a protection level, but the other is that it can be a parameter that arises from data that governs the maximum/minimum demand realizations.

4.2 Two Items

Building upon the results obtained in the previous section, we examine the setting with multiple items, first considering the two item case. To begin our analysis of the two item case, we first consider a case with two identical items which share identical cost and demand parameters under the assumption that $\delta^L = \delta^U$. We subsequently relax this assumption and then extend the analysis to two non-identical items where cost and demand parameters may vary by item.

Recall that for ease of notation, we define the nature's budget of uncertainty to be

$$C = \mathbb{E}\left[\tilde{Z}^+\right] + \delta_Z\sqrt{\mathrm{Var}\left(\tilde{Z}^+\right)} = \left|\mathbb{E}\left[\tilde{Z}^-\right]\right| + \delta_Z\sqrt{\mathrm{Var}\left(\tilde{Z}^-\right)}.$$

We will continue to use this representation throughout the remainder of this paper. Note that this differs from the previously defined c.

4.2.1 Two Identical Items

Suppose that we have two items indexed by i with identical demand parameters μ, δ, and σ as well as identical backorder and holding costs b and h. Then by symmetry, each of the two items must have the same inventory level.

Recall that in the single item case, only a single constraint could be active for either ε^+ or ε^-. But when we consider the two item case, two constraints will be active. Which two constraints are active will be determined by the budget given to nature. In this two identical item setting, the uncertainty set will be as follows:

$$U(\delta, \delta_Z) = \left\{ \boldsymbol{\varepsilon} : \begin{array}{cc} \sum_{i=1}^{2} \varepsilon_i^+ \sigma_i \leq C^+ & \\ \sum_{i=1}^{2} \varepsilon_i^- \sigma_i \leq C^- & \\ 0 \leq \varepsilon_i^+ \leq \delta & , i = 1, 2 \\ 0 \leq \varepsilon_i^- \leq \delta & , i = 1, 2 \\ \varepsilon_i = \varepsilon_i^+ - \varepsilon_i^- & , i = 1, 2 \\ \varepsilon_i^+, \varepsilon_i^- \geq 0 & , i = 1, 2 \end{array} \right\}. \tag{4.1}$$

To begin our analysis, we must look at a few cases. Observe that nature's budget of uncertainty C must lie within one of three intervals which partition the non-negative real numbers: $[0, \sigma\delta), [\sigma\delta, 2\sigma\delta), [2\sigma\delta, \infty)$. That is, for each of the joint constraints, C must lie within one of these three intervals. We first begin by studying the problem facing nature. Recall that nature selects the realization of demand by selecting a perturbation from the mean. In this way, nature is choosing the end of period stock levels for each item within a bounded set. We begin by examining the dependence of the feasible region on the value of the budget of uncertainty C. Suppose that $\delta = 2$ and $\sigma = 1$. The graph of the dependence of the feasible region for $\boldsymbol{\varepsilon}$ with respect to C is displayed in Figure 3.

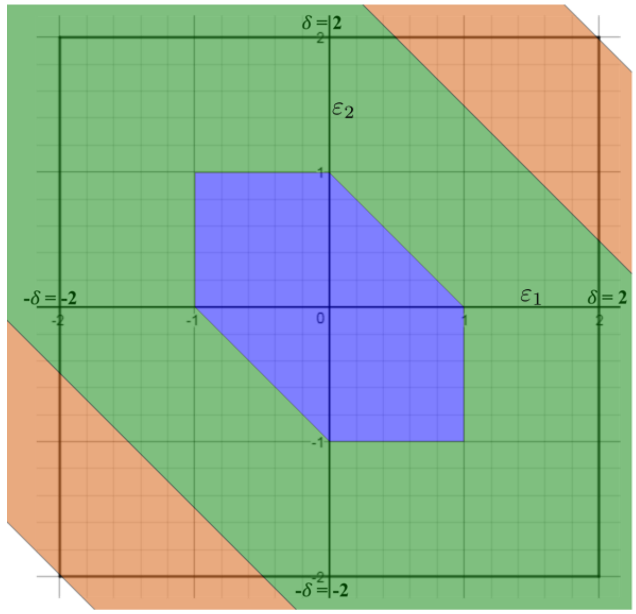

Figure 3: Depiction of feasible region for nature under a variety of budgets of uncertainty with $\delta = 2, \sigma = 1$, and: a) $C = 1$ in blue, b) $C = 2.5$ in green, c) $C = 4$ in orange.

The constraints involving δ form the square area depicted in black while the joint budget constraints form the hexagonal areas in color. Then for each of these cases, the feasible region for nature is the intersection of the two areas. Note that for $C = 1$, the colored area (blue) does not intersect the black lines. Then neither ε_1 (x-axis) nor ε_2 (y-axis) can attain a value of δ. As the budget grows to $C = 2.5$, δ can be attained, but ε_1 and ε_2 cannot simultaneously attain δ. That is, while actions $(\varepsilon_1, \varepsilon_2) \in \{(\delta, 0), (-\delta, 0), (\delta, -\delta), (0, \delta), (0, -\delta), (-\delta, \delta)\}$ are feasible, points such as (δ, δ) and $(-\delta, -\delta)$ are not yet feasible. Eventually, we see that the black square ($[-2, 2] \times [-2, 2]$) is fully contained within the hexagonal area of the joint budget constraints as C increases. At this point, C has become large enough so that the intersection of the two areas is simply the square, and we see that the joint budget constraints are not binding. Therefore, for sufficiently large C, the problem separates by item.

In general there will be six extreme points, unless C is sufficiently large so that the joint constraints will never be active. The six extreme points correspond to the following actions taken by nature. Each action can be a positive or negative deviation from the mean. Additionally, when both actions are in the same direction, that is both positive or negative deviations, we obtain two extreme points due to the order in which the budgets are utilized. i.e. $\varepsilon_1, \varepsilon_2 > 0$ is not sufficient to determine an extreme point as there should be two different extreme points, corresponding to the item which receives the larger portion of the budget. This can be seen by the diagonal line in the upper-right and bottom-left quadrants in Figure 3, which can be contrasted with the single extreme point in the upper-left and bottom-right. More generally, we can compute the possible extreme points by choosing which constraints will be active. The six general extreme points are as follows:

$$
\begin{aligned}
&(i)\ \left(-\left(\frac{C}{\sigma}\wedge\delta\right), -\left[\left(\frac{C-\sigma\delta}{\sigma}\right)^{+}\wedge\delta\right]\right), \qquad &&(iv)\ \left(\left[\left(\frac{C-\sigma\delta}{\sigma}\right)^{+}\wedge\delta\right], \left(\frac{C}{\sigma}\wedge\delta\right)\right),\\
&(ii)\ \left(-\left[\left(\frac{C-\sigma\delta}{\sigma}\right)^{+}\wedge\delta\right], -\left(\frac{C}{\sigma}\wedge\delta\right)\right), \qquad &&(v)\ \left(-\left(\frac{C}{\sigma}\wedge\delta\right), \left(\frac{C}{\sigma}\wedge\delta\right)\right), \text{ and}\\
&(iii)\ \left(\left(\frac{C}{\sigma}\wedge\delta\right), \left[\left(\frac{C-\sigma\delta}{\sigma}\right)^{+}\wedge\delta\right]\right), \qquad &&(vi)\ \left(\left(\frac{C}{\sigma}\wedge\delta\right), -\left(\frac{C}{\sigma}\wedge\delta\right)\right).
\end{aligned}
\tag{4.2}
$$

Now that we have enumerated the extreme points, we return to nature's problem of choosing the end of period stock levels. Recall that nature perturbs the demand within the convex region described by the extreme points listed above in order to maximize the cost to the seller. This is depicted in Figure 4 when the stock level of each item is equal to its mean.

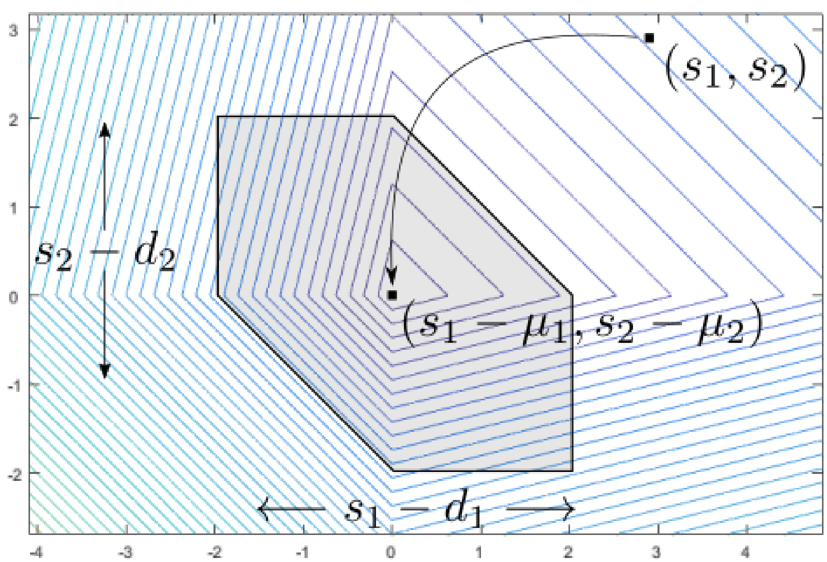

Figure 4: Depiction of cost vs end of period inventory levels for two identical items when $s = \mu$. $b = 4, h = 1, \mu = 3, \sigma = 1, \delta = 2, C = 2$.

In the contour plot, each line connects points with identical costs. Clearly the cost is minimal to the seller if the ending inventory is 0 for both items. Hence the costs are increasing as the end of period inventory moves away from the point (0, 0). Note that the distance between the isocost lines varies by quadrant. This is due to the fact that the backorder cost is significantly larger than the holding cost. Note that in the positive quadrant (top right), both items experience holding costs while in the negative quadrant (bottom left), both items experience backorder costs. In the other two quadrants, one item experiences a backorder cost while the other item experiences a holding cost.

In the example found in Figure 4, we see that the two extreme points corresponding to $(\delta, -\delta)$ and $(-\delta, \delta)$ result in the largest cost to the seller and hence are optimal for nature. However, note that if the seller were to increase both inventory levels, the total cost to the seller could be reduced. This is depicted in Figure 5 when the seller has chosen to stock each item optimally.

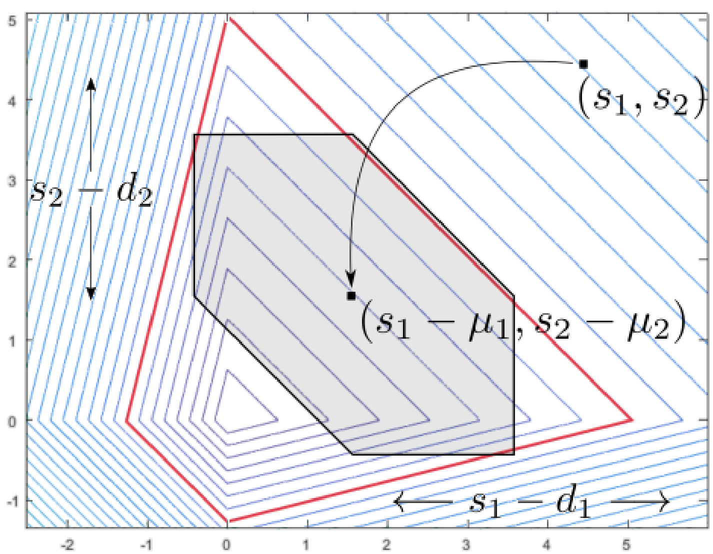

Figure 5: Depiction of cost vs end of period inventory levels for two identical items when $s = s^*$. $b = 4, h = 1, \mu = 3, \sigma = 1, \delta = 2, C = 2$.

Note that the stock level has increased from $(3, 3)$ in order to make nature indifferent among four different actions which result in equal cost to the seller (shown in red). We utilize this idea of indifference among actions in the subsequent sections to guide our analysis.

We now begin our analysis of the three intervals $[0, \sigma\delta), [\sigma\delta, 2\sigma\delta), [2\sigma\delta, \infty)$ in which C might exist.

First, consider the case for which the budget $C < \sigma\delta$, corresponding to the blue region in Figure 3. In this scenario, nature cannot perturb either item to its extreme since there is not sufficient budget. Then the demand chosen by nature is bounded above by $\mu + C$ and bounded below by $\mu - C$. Recall, due to symmetry, both items have stock levels equal to a common value, s. Let

$$f(s) = \max_{\varepsilon_1, \varepsilon_2 \in U} \{\max\{b(\mu + \varepsilon_1\sigma - s), h(s - \mu - \varepsilon_1\sigma)\} + \max\{b(\mu + \varepsilon_2\sigma - s), h(s - \mu - \varepsilon_2\sigma)\}\}$$

which represents the cost associated with choosing a beginning of period stock level of s. Note that $f(s)$ is a function of the beginning of period inventory, which is contrasted with the end of period inventory depicted in Figures 4 and 5. We depict $f(s)$ which in Figure 6

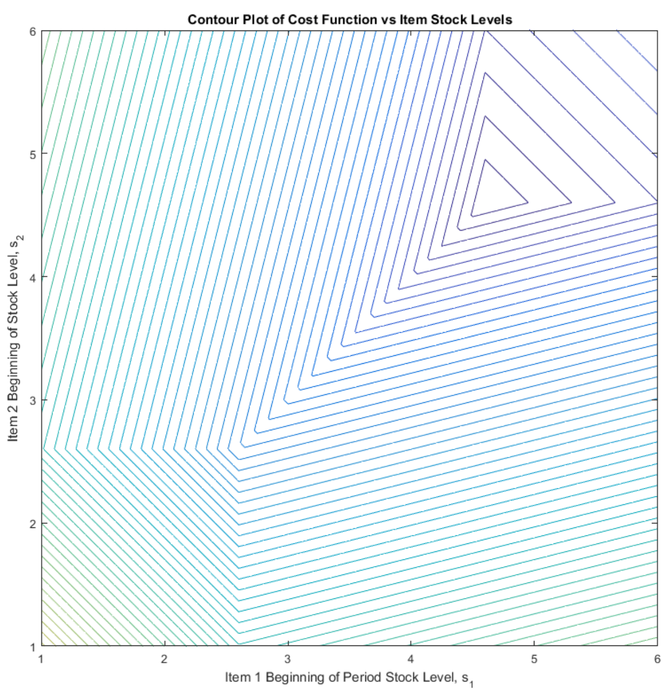

Figure 6: Contour plot of cost vs beginning of period stock level for a two identical item case with $h = 1, b = 4, \mu = 3, \sigma = 1, \delta = 2, C = 2$.

Each line in the contour plot connects points with identical costs, $f(s)$. In this contour plot, the contour lines indicate a single minimum near $(5, 5)$ with costs increasing as stock levels move away from this point. Note that the plot seems to indicate that all points within the smallest triangle are equivalent. However, this is not the case but is due to the resolution of the contour plot. Observe that as before, the contour plot seems to split into four regions, each corresponding to a different action taken by nature. This can be best understood by looking at the stock levels set for each item. The regions can be gathered into the following sections, moving clockwise from the bottom left: a) Low stock level for both items, b) low stock level for item 1, high stock level for item 2, c) high stock level for both items, and d) high stock level for item 1, low stock level for item 2. Let us summarize these regions briefly.

For region a), the stock levels of both items are both extremely low. In particular, these items are stocked below the mean. In this regime, both items will experience backorder costs with additional backorders imposed by nature. Next, for region b), item 1 has relatively low stock while item 2 has relatively high stock. Therefore, nature associates backorder costs with item 1 and holding costs with item 2. In region c), we see that both items have large stock levels. In this regime, nature associates holding costs with both items, as the stock level is high enough to prevent a large number of backorders. Finally, in region d), because item 1 has a high stock level while item 2 has a low stock level, nature will associate holding costs with item 1 and backorder costs with item 2.

Note that the spacing between the contour lines differs among these regions. That is, the spacing between contour lines is quite small for region a), slightly larger for regions b) and d), and quite large for region c). This is driven by the large difference between the backorder cost $b = 4$ and the holding cost $h = 1$. Note that the symmetry of our problem will only permit optimal solutions to lie on the diagonal, where both items have the same stock level. Then the plot can be simplified as in Figure 7.

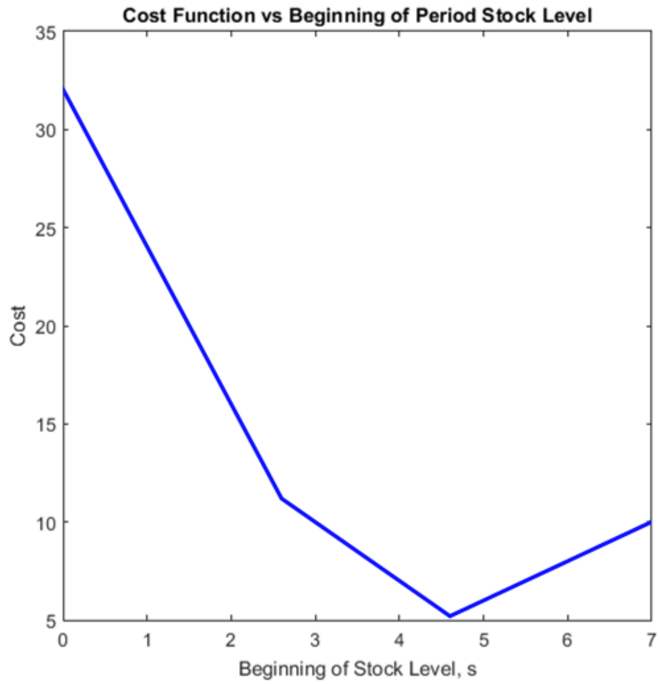

Figure 7: Simplified plot of cost function for a two identical item case with $h = 1, b = 4, \mu = 3, \sigma = 1, \delta = 2, C = 2$.

Recall that there was a stark difference in spacing between contour lines in the four regions of the preceding contour plot. This can also be observed in this two-dimensional plot. Observe that the cost changes rapidly through the first segment of the plot as the stock level s increases from 0 to roughly 2 at a slope of -8. This corresponds to region a) in our contour plot. Next, observe that the decrease slows to a slope of -3 as the stock level increases from around 2 to slightly less than 5. This segment corresponds to entering the boundary of regions b) and d) in the contour plot. Finally, observe that once the minimum value is reached, the rate of change is much smaller, with a slope of only 2. An observation that can be made from this graph is that the consequences of having too little inventory are much higher than those of having too much inventory. This is to be expected as the backorder cost is significant higher than the holding cost.

Returning to our analysis of the case where $C < \sigma\delta$, our approach is as follows:

Step 1 We propose a candidate solution s^*.

Step 2 We find $f(s^*)$ by analyzing the set of extreme points that can be selected by nature which are the worst case for the given s^*.

Step 3 We demonstrate that the set of extreme points chosen by nature imply that any deviation from s^* is suboptimal to the seller and hence s^* is optimal.

We will continue to utilize this approach through the end of Section 4.2.4. Now, let

$$s^* = \mu + \frac{b}{b+h}C$$

and let

$$g\left(s, \boldsymbol{\varepsilon}\right) = \max\left\{b(\mu + \varepsilon_1\sigma - s), h(s - \mu - \varepsilon_1\sigma)\right\} + \max\left\{b(\mu + \varepsilon_2\sigma - s), h(s - \mu - \varepsilon_2\sigma)\right\}$$

represent the cost for a particular stock level, s, chosen by the seller and action, $\boldsymbol{\varepsilon}$, chosen by nature. Recall that from (4.2), we know that the feasible region has the following six extreme points:

$$(\varepsilon_1, \varepsilon_2) \in \left\{\left(\frac{-C}{\sigma}, 0\right), \left(\frac{-C}{\sigma}, \frac{C}{\sigma}\right), \left(0, \frac{-C}{\sigma}\right), \left(\frac{C}{\sigma}, \frac{-C}{\sigma}\right), \left(\frac{C}{\sigma}, 0\right), \left(0, \frac{C}{\sigma}\right)\right\}.$$

We will demonstrate that at $s = s^*$, nature is indifferent among the four below extreme points and the remaining two extreme points are strictly worse for nature. The four extreme points for which nature is indifferent are:

$$(\varepsilon_1, \varepsilon_2) \in \left\{\left(\frac{-C}{\sigma}, 0\right), \left(\frac{-C}{\sigma}, \frac{C}{\sigma}\right), \left(0, \frac{-C}{\sigma}\right), \left(\frac{C}{\sigma}, \frac{-C}{\sigma}\right)\right\}.$$

That is,

$$g\left(s^*, \left(\frac{-C}{\sigma}, 0\right)\right) = g\left(s^*, \left(\frac{-C}{\sigma}, \frac{C}{\sigma}\right)\right) = g\left(s^*, \left(0, \frac{-C}{\sigma}\right)\right) = g\left(s^*, \left(\frac{C}{\sigma}, \frac{-C}{\sigma}\right)\right).$$

For each of the four actions, $g\left(s^*, \boldsymbol{\varepsilon}\right) = 2h(s^* - \mu) + hC$. This can be seen easily for the cases of $\left(\frac{-C}{\sigma}, 0\right)$ and $\left(0, \frac{-C}{\sigma}\right)$ which associate holding costs for both items, but is not immediately obvious for the other two cases which associate a holding cost with one item and a backorder cost to the second item. But consider the cost contribution of the item i with $\varepsilon_i = 0$ which experiences a demand of μ. It has an associated cost of $h(s^* - \mu) = \frac{bh}{b+h}C$. Now consider the cost when the item experiences a demand of $\mu + C$. Then the cost contribution is

$$\begin{aligned} b(\mu + C - s^*) &= bC - b\frac{b}{b+h}C \\ &= bC\left(\frac{b+h-b}{b+h}\right) \\ &= \frac{bh}{b+h}C. \end{aligned}$$

Suppose that s^* were not optimal and that the optimal s lies above s^*. But for $s > s^*$, nature can choose action $\left(\frac{-C}{\sigma}, 0\right)$ or $\left(0, \frac{-C}{\sigma}\right)$ to inflict a cost of $f(s) = 2h(s-\mu) + hC > f(s^*)$. Now suppose that the optimal $s = s^* - \eta, \eta > 0$. But observe that nature can then choose action $\left(\frac{-C}{\sigma}, \frac{C}{\sigma}\right)$ or $\left(\frac{C}{\sigma}, \frac{-C}{\sigma}\right)$ to inflict a cost of

$$\begin{aligned} f(s) = h(s - \mu + C) + b(\mu + C - s) &= h(s^* - \mu + C - \eta) + b(\mu + C - (s^* - \eta)) \\ = h(s^* - \mu + C) + b(\mu + C - s^*) + (b-h)\eta &= f(s^*) + (b-h)\eta > f(s^*). \end{aligned}$$

Recall, however, that the feasible region for this problem has six extreme points. The two remaining extreme points are $(\varepsilon_1, \varepsilon_2) = \left(\frac{C}{\sigma}, 0\right)$ and $(\varepsilon_1, \varepsilon_2) = \left(0, \frac{C}{\sigma}\right)$. These extreme points are suboptimal for nature which can be easily seen by comparison to extreme points $\left(\frac{C}{\sigma}, \frac{-C}{\sigma}\right)$ and $\left(\frac{-C}{\sigma}, \frac{C}{\sigma}\right)$. The difference is that under $\left(0, \frac{C}{\sigma}\right)$, the first item experiences a demand of μ while under $\left(\frac{-C}{\sigma}, \frac{C}{\sigma}\right)$, the first item experiences a demand of $\mu - C$. Since $s^* > \mu$, this item experiences a holding cost, which will be larger for $\left(\frac{-C}{\sigma}, \frac{C}{\sigma}\right)$.

Now consider the second case where $\sigma\delta \leq C < 2\sigma\delta$, which corresponds to the green region in Figure 3. Nature now has a large enough budget to perturb a single item's demand by up to $\sigma\delta$. However, nature cannot yet perturb both item's demands independently. That is, if nature chooses to increase the demand for item 1 to $\mu+\sigma\delta$, this will cause the joint constraint to limit the maximum demand that nature can select for item 2. A similar statement is true for decreasing the demands. A natural observation is that since the budget of nature has increased, nature's ability to do harm should also increase, resulting in a larger cost to the seller.

Similar to the analysis above, we consider an inventory level s^* and a set of extreme points among which nature is indifferent. Let

$$s^* = \mu + \sigma\delta - \frac{h}{b+h}C.$$

Then under nature's new budget, the new set of extreme points is

$$(\varepsilon_1, \varepsilon_2) \in \left\{ \left(-\delta, \tfrac{-(C-\sigma\delta)}{\sigma}\right), (-\delta, \delta), \left(\tfrac{-(C-\sigma\delta)}{\sigma}, -\delta\right), (\delta, -\delta), \left(\delta, \tfrac{(C-\sigma\delta)}{\sigma}\right), \left(\tfrac{(C-\sigma\delta)}{\sigma}, \delta\right) \right\}.$$

As before, we will show that the cost for a set of four actions is identical with the remaining two actions suboptimal for nature. Observe that the cost under actions $(\varepsilon_1, \varepsilon_2) \in \left\{ \left(-\delta, \tfrac{-(C-\sigma\delta)}{\sigma}\right), (-\delta, \delta), \left(\tfrac{-(C-\sigma\delta)}{\sigma}, -\delta\right), (\delta, -\delta) \right\}$ is identical for $s = s^*$ with total cost:

$$f(s^*) = 2h\left(\sigma\delta - \frac{h}{b+h}C\right) + hC.$$

As before, this is clear for $(\varepsilon_1, \varepsilon_2) \in \left(-\delta, \tfrac{-(C-\sigma\delta)}{\sigma}\right), \left(\tfrac{-(C-\sigma\delta)}{\sigma}, -\delta\right)$, and we show the cost for the other two extreme points below.

$$\begin{aligned}
g\left(s^*, (-\delta, \delta)\right) &= h(s^* - (\mu - \sigma\delta)) + b(\mu + \sigma\delta - s^*) \\
&= h\left(\left(\mu + \sigma\delta - \frac{h}{b+h}C\right) - (\mu - \sigma\delta)\right) + b\left(\mu + \sigma\delta - \left(\mu + \sigma\delta - \frac{h}{b+h}C\right)\right) \\
&= 2h\sigma\delta - \frac{h^2}{b+h}C + \frac{bh}{b+h}C \\
&= 2h\sigma\delta - 2\frac{h^2}{b+h}C + \frac{bh}{b+h}C + \frac{h^2}{b+h}C \\
&= 2h\left(\sigma\delta - \frac{h}{b+h}C\right) + \frac{h(b+h)}{b+h}C \\
&= 2h\left(\sigma\delta - \frac{h}{b+h}C\right) + hC.
\end{aligned}$$

Now we show that s^* is optimal by considering the cost for other values of s. If the seller were to select $s > s^*$, actions $\left(-\delta, \frac{-(C-\sigma\delta)}{\sigma}\right)$ and $(\frac{-(C-\sigma\delta)}{\sigma}, -\delta)$ result in a higher cost:

$$\begin{aligned} g\left(s, \left(-\delta, \frac{-(C-\sigma\delta)}{\sigma}\right)\right) &= g\left(s, \left(\frac{-(C-\sigma\delta)}{\sigma}, -\delta\right)\right) \\ &= h(s - (\mu - \sigma\delta)) + h(s - (\mu - (C - \sigma\delta))) \\ &= g\left(s^*, \left(-\delta, \frac{-(C-\sigma\delta)}{\sigma}\right)\right) + 2h(s - s^*) \\ &> g\left(s^*, \left(-\delta, \frac{-(C-\sigma\delta)}{\sigma}\right)\right). \end{aligned}$$

If the seller were to select $s < s^*$, actions $(-\delta, \delta)$ and $(\delta, -\delta)$ result in higher cost:

$$\begin{aligned} g\left(s, \left(\delta, \frac{(C-\sigma\delta)}{\sigma}\right)\right) &= g\left(s, \left(\frac{(C-\sigma\delta)}{\sigma}, \delta\right)\right) \\ &= b((\mu + \sigma\delta) - s) + b((\mu - (C - \sigma\delta)) - s) \\ &= g\left(s^*, \left(-\delta, \frac{-(C-\sigma\delta)}{\sigma}\right)\right) + 2b(s^* - s) \\ &> g\left(s^*, \left(-\delta, \frac{-(C-\sigma\delta)}{\sigma}\right)\right). \end{aligned}$$

Again, we have two remaining extreme points to analyze: $(\varepsilon_1, \varepsilon_2) = \left(\delta, \frac{(C-\sigma\delta)}{\sigma}\right)$ and $(\varepsilon_1, \varepsilon_2) = \left(\frac{(C-\sigma\delta)}{\sigma}, \sigma\right)$. We demonstrate that these are suboptimal for nature when $s = s^*$. Observe that

$$\begin{aligned} g\left(s^*, \left(\delta, \frac{(C-\sigma\delta)}{\sigma}\right)\right) &= g\left(s^*, \left(\frac{(C-\sigma\delta)}{\sigma}, \sigma\right)\right) \\ =& b((\mu + C - \sigma\delta) - s^*) + b(\mu + \sigma\delta - s^*) \\ =& b\left(\mu + C - \sigma\delta - \left(\mu + \sigma\delta - \frac{h}{b+h}C\right)\right) + b\left(\mu + \sigma\delta - \left(\mu + \sigma\delta - \frac{h}{b+h}C\right)\right) \\ =& b\left(C - 2\sigma\delta + 2\frac{h}{b+h}C\right). \end{aligned}$$

Comparing the above expression to the cost under action $(\varepsilon_1, \varepsilon_2) = \left(-\delta, \frac{-(C-\sigma\delta)}{\sigma}\right)$, $g\left(s^*, \left(-\delta, \frac{-(C-\sigma\delta)}{\sigma}\right)\right)$, we wish to show that

$$b\left(C - 2\sigma\delta + 2\frac{h}{b+h}C\right) < 2h\left(\sigma\delta - \frac{h}{b+h}C\right) + hC$$

Recall that it was assumed that $C < 2\sigma\delta$. Then we have that

$$\begin{aligned}
C &< 2\sigma\delta \\
(b+h)C &< 2(b+h)\sigma\delta \\
bC - 2b\sigma\delta &< 2h\sigma\delta - hC \\
bC - 2b\sigma\delta + 2hC &< 2h\sigma\delta + hC \\
bC - 2b\sigma\delta + 2\frac{h(b+h)}{b+h}C &< 2h\sigma\delta + hC \\
bC - 2b\sigma\delta + 2\frac{bh}{b+h}C &< 2h\sigma\delta - 2\frac{h^2}{b+h}C + hC \\
b\left(C - 2\sigma\delta + 2\frac{h}{b+h}C\right) &< 2h\left(\sigma\delta - \frac{h}{b+h}C\right) + hC.
\end{aligned}$$

Finally, suppose that $C > 2\sigma\delta$ which can be seen in orange in Figure 3. That is, nature has a relatively large budget. This budget is sufficiently large that nature can raise or lower the demand for each item to either extreme and hence, the joint budget constraint can never be binding. As a result, this problem separates into two independent single item problems and the optimal solution is that of the single item case, $s^* = \mu + \frac{b-h}{b+h}\sigma\delta$.

We summarize the previously proven results with the following proposition and corollary:

Proposition 4.2 *For the cases described, the optimal inventory level is:*

(i) For $0 \leq C < \sigma\delta, s^* = \mu + \frac{b}{b+h}C,\ f(s^*) = 2h\left(\frac{b}{b+h}C\right) + hC.$

(ii) For $\sigma\delta \leq C < 2\sigma\delta, s^* = \mu + \sigma\delta - \frac{h}{b+h}C,\ f(s^*) = 2h\left(\sigma\delta - \frac{h}{b+h}C\right) + hC.$

(iii) For $2\sigma\delta < C, s^* = \mu + \frac{b-h}{b+h}\sigma\delta,\ f(s^*) = 2h\left(\frac{b-h}{b+h}\sigma\delta\right) + h(2\sigma\delta).$

Corollary 4.3 *The optimal inventory level and optimal cost can be expressed as*

$$\begin{aligned}
s^* &= \mu + \frac{b}{b+h}(C \wedge \sigma\delta) - \frac{h}{b+h}\left[(C-\sigma\delta)^+ \wedge (\sigma\delta)\right] \\
f(s^*) &= 2h(s^*-\mu) + h(C \wedge 2\sigma\delta) \\
&= 2h\left(\frac{b}{b+h}(C \wedge \sigma\delta) - \frac{h}{b+h}\left[(C-\sigma\delta)^+ \wedge (\sigma\delta)\right]\right) + h(C \wedge 2\sigma\delta)
\end{aligned}$$

Observe that the cost is non-decreasing in the budget C as one would expect. However, while one might also expect the inventory levels to be nondecreasing, this is not necessarily the case. Over the first interval $[0, \sigma\delta)$, the value of s^* increases from μ to $\mu + \frac{b}{b+h}\sigma\delta$. As nature's budget begins to increase, so too does the inventory in order to better protect against backorders. Then in the second interval $[\sigma\delta, 2\sigma\delta)$, the value of s^* decreases from $\mu + \frac{b}{b+h}\sigma\delta$ to $\mu + \frac{b-h}{b+h}\sigma\delta$. In this interval, nature has a sufficiently large budget so that the additional holding costs that can be imposed by nature also become large. In order to protect against this as well, the inventory level begins to decrease. For C larger than $2\sigma\delta$, the value of s^* does not change. Note that this is strikingly different than in the stochastic model. In the stochastic model, a larger value of C would correspond to more variability, and hence an increased variance for the demand process. Recall that the classical newsvendor problem has an optimal solution at $F^{-1}\left(\frac{b}{b+h}\right)$, where F^{-1} denotes the inverse cumulative distribution function of the demand process. Under a demand process with the same mean but higher variance, this would necessarily increase the optimal inventory level.

Note from the preceding analysis that the optimal inventory level is also a function of the cost parameters, b and h. In particular, the ratio of costs, b/h, determines the extent to which the optimal inventory level exceeds the mean. As the ratio goes to infinity, the inventory level s^* also increases so as to make nature impotent. Conversely, as the ratio approaches 1, s^* decreases towards the mean when $C > 2\sigma\delta$. This second observation is due to the symmetric nature of the uncertainty set. That is, $\delta^L = \delta^U$. Should these bounds be different in magnitude, then the optimal inventory level may not converge to the mean as

b/h approaches 1.

Another observation that can be made is that the form of the costs suggests that $f(\boldsymbol{s})$ has two separate components: 1) the cost associated with the inventory level and 2) the additional cost imposed by nature. As we have already observed, if the seller chooses a sufficiently large stocking level, nature will draw solely upon the budget for negative deviations, choosing only smaller than expected demands. That is, for an inventory level of $s > \mu + \frac{b}{b+h}\sigma\delta$ observe that

$$\begin{aligned} b(\mu + \sigma\delta - s) &< b\left(\mu + \sigma\delta - \left(\mu + \frac{b}{b+h}\sigma\delta\right)\right) \\ &= b\left(\frac{h}{b+h}\sigma\delta\right) \\ &= h\left(\frac{b}{b+h}\sigma\delta\right) \\ &= h\left(\mu + \frac{b}{b+h}\sigma\delta - \mu\right) \\ &< h\left(s - \mu\right). \end{aligned}$$

Then for such a large level of inventory, nature would prefer to do nothing than to choose a large demand. In this regime, it is suboptimal to have $\varepsilon^+ > 0$ for any item i, and nature would simply exhaust the budget for negative deviations (ε^-). Assuming that $C < 2\sigma\delta$, this would result in a total amount of deviation of C. That is, $\sum_{i=1}^{2} |d_i - \mu| = C$ and nature has added an incremental cost of hC. Here, the inventory cost is $2h(s - \mu)$ and the penalty added by nature is hC.

Similarly, if the seller chooses a sufficiently small stocking level, then nature will draw solely upon the budget for positive deviations, choosing only larger than expected demands. Similar to the case above, this corresponds to a regime where $s < \mu - \frac{h}{b+h}\sigma\delta$ so that:

$$
\begin{aligned}
h(s-\mu+\sigma\delta) &< h\left(\mu - \frac{h}{b+h}\sigma\delta - \mu + \sigma\delta\right) \\
&= h\left(\frac{b}{b+h}\sigma\delta\right) \\
&= b\left(\frac{h}{b+h}\sigma\delta\right) \\
&= b\left(\mu + \frac{h}{b+h}\sigma\delta - \mu\right) \\
&= b\left(\mu - \left(\mu - \frac{h}{b+h}\sigma\delta\right)\right) \\
&< b\left(\mu - s\right).
\end{aligned}
$$

Thus, for such a low level of inventory, nature would prefer $\varepsilon = 0$ over $\varepsilon < 0$. In this regime, nature would choose a total amount of deviation of C by fully utilizing the budget for large demand, and again, $\sum_{i=1}^{2} |d_i - \mu| = C$. In this setting, the inventory cost is $b(\mu - s)$ and the cost added by nature is bC.

However, if the seller chooses an intermediate stocking level, nature may have an incentive to draw from both budgets. In this regime, it may be profitable for nature to choose a large demand for one item and choose a small demand for the second item. In this case, the total deviation is much larger than the other two cases. Consider stocking each item at its mean, $s = \mu$. Then for this set of stock levels, nature would seek to consume both budgets fully. Assuming that $C > \sigma\delta$, this results in a total deviation of $\sum_{i=1}^{2} |d - \mu| = 2\sigma\delta$ with no inventory cost, but with an incremental cost of $(b + h)\sigma\delta$ added by nature.

Note that in this last setting with $s = \mu$, the incremental cost added by nature is larger than in either of the preceding two settings. Stated differently, while the seller stocks a more "appropriate" inventory level, nature has more power to inflict additional costs. In the other two cases, the seller pays a higher cost by stocking further from the mean in order to remove nature's incentive to draw upon both budgets. This can be thought of as paying a "price" to induce less uncertainty.

The previous analysis was performed under the assumption that the uncertainty set was symmetric, that is, $\varepsilon_i^- \leq \delta, \varepsilon_i^+ \leq \delta \; \forall \, i$. While this may be true in some instances, this will not be the case for all situations. However, our model can also accommodate a non-symmetric uncertainty set, as we now see.

In the following section, we extend the problem formulation to encompass non-symmetric uncertainty sets. That is, there exist separate bounds on $\varepsilon_i^-, \varepsilon_i^+$ so that $\varepsilon_i^- \leq \delta^L$ and $\varepsilon_i^+ \leq \delta^U$. While we analyzed the problem in this section for any possible budget of uncertainty C, we now assume C is chosen to be sufficiently large. In this setting, we will make the assumption that $C > \max\left\{\sigma\delta^L, \sigma\delta^U\right\}$. This value of C does not detract from the analysis, and presents a more realistic view of the problem. That is to say, if C were relatively small, then nature would not be able to inflict much harm, which would correspond to a low variance demand process. In this setting, the demand process can easily be approximated by a stochastic model. Thus, in the spirit of robust modeling and analysis, we assume that the demand process is highly variable, and C is relatively large throughout the remainder of this paper. However, the precise definition of sufficiently large will change depending on the problem setting, and will be redefined as needed.

4.2.2 Non-symmetric Uncertainty Sets

Let us now allow the bounds on ε_i^- and ε_i^+ to differ while demand parameters $\delta^L, \delta^U, \mu, \sigma$ remain identical for both items. Then this corresponds to the following uncertainty set

$$U\left(\boldsymbol{\delta}, \delta_Z\right) = \left\{ \boldsymbol{\varepsilon} : \begin{array}{c} \sum_{i=1}^{2} \varepsilon_i^+ \sigma \leq \mathbb{E}\left[\tilde{Z}^+\right] + \delta_Z \sqrt{\mathrm{Var}\left(\tilde{Z}^+\right)} \\ \sum_{i=1}^{2} \varepsilon_i^- \sigma \leq \left|\mathbb{E}\left[\tilde{Z}^-\right]\right| + \delta_Z \sqrt{\mathrm{Var}\left(\tilde{Z}^-\right)} \\ 0 \leq \varepsilon_i^+ \leq \delta^U \ \forall\ i \\ 0 \leq \varepsilon_i^- \leq \delta^L \ \forall\ i \\ \varepsilon_i = \varepsilon_i^+ - \varepsilon_i^- \ \forall\ i \\ \varepsilon_i^+, \varepsilon_i^- \geq 0 \ \forall\ i \end{array} \right\}. \tag{4.3}$$

Since the bounds on ε_i^+ and ε_i^- may differ, the planner gains additional flexibility in modeling the demand process.

The approach from the previous section can still be applied, though we now have several more intervals to analyze. Before we begin, note that there are four possible scenarios associated with the asymmetry:

$$\begin{array}{rrl} \text{(i)} & 2\delta^U \leq \delta^L & , \\ \text{(ii)} & \delta^U \leq \delta^L & , \\ & 2\delta^U > \delta^L & , \\ \text{(iii)} & 2\delta^L \leq \delta^U & , \text{ and} \\ \text{(iv)} & \delta^L \leq \delta^U & , \\ & 2\delta^L > \delta^U & . \end{array}$$

Depending on which scenario we are in, we will have the following differing intervals that partition the non-negative real numbers:

$$\begin{aligned}
&\text{(i)} \quad [0,\sigma\delta^U),[\sigma\delta^U,2\sigma\delta^U),[2\sigma\delta^U,\sigma\delta^L),[\sigma\delta^L,2\sigma\delta^L),[2\sigma\delta^L,\infty)\\
&\text{(ii)} \quad [0,\sigma\delta^U),[\sigma\delta^U,\sigma\delta^L),[\sigma\delta^L,2\sigma\delta^U),[2\sigma\delta^U,2\sigma\delta^L),[2\sigma\delta^L,\infty)\\
&\text{(iii)} \quad [0,\sigma\delta^L),[\sigma\delta^L,2\sigma\delta^L),[2\sigma\delta^L,\sigma\delta^U),[\sigma\delta^U,2\sigma\delta^U),[2\sigma\delta^U,\infty)\\
&\text{(iv)} \quad [0,\sigma\delta^L),[\sigma\delta^L,\sigma\delta^U),[\sigma\delta^U,2\sigma\delta^L),[2\sigma\delta^L,2\sigma\delta^U),[2\sigma\delta^U,\infty).
\end{aligned}$$

However, as was the case in (4.2), the six possible extreme points of nature's actions can be expressed independently of the orderings of the uncertainty parameters. We enumerate these extreme points in their general forms below:

$$\begin{aligned}
&\text{(i)}\ \left(-\left(\tfrac{C}{\sigma}\wedge\delta^L\right),-\left[\left(\tfrac{C-\sigma\delta^L}{\sigma}\right)^+\wedge\delta^L\right]\right), &&\text{(iv)}\ \left(\left[\left(\tfrac{C-\sigma\delta^U}{\sigma}\right)^+\wedge\delta^U\right],\left(\tfrac{C}{\sigma}\wedge\delta^U\right)\right),\\
&\text{(ii)}\ \left(-\left[\left(\tfrac{C-\sigma\delta^L}{\sigma}\right)^+\wedge\delta^L\right],-\left(\tfrac{C}{\sigma}\wedge\delta^L\right)\right), &&\text{(v)}\ \left(-\left(\tfrac{C}{\sigma}\wedge\delta^L\right),\left(\tfrac{C}{\sigma}\wedge\delta^U\right)\right),\ \text{and}\\
&\text{(iii)}\ \left(\left(\tfrac{C}{\sigma}\wedge\delta^U\right),\left[\left(\tfrac{C-\sigma\delta^U}{\sigma}\right)^+\wedge\delta^U\right]\right), &&\text{(vi)}\ \left(\left(\tfrac{C}{\sigma}\wedge\delta^U\right),-\left(\tfrac{C}{\sigma}\wedge\delta^L\right)\right).
\end{aligned}$$

Because of our previous assumption that $C > \max\left\{\sigma\delta^L, \sigma\delta^U\right\}$, we are only interested in the following intervals of the above partitions:

$$\begin{aligned}
&\text{(i)} \quad [\sigma\delta^L,2\sigma\delta^L),[2\sigma\delta^L,\infty)\\
&\text{(ii)} \quad [\sigma\delta^L,2\sigma\delta^U),[2\sigma\delta^U,2\sigma\delta^L),[2\sigma\delta^L,\infty)\\
&\text{(iii)} \quad [\sigma\delta^U,2\sigma\delta^U),[2\sigma\delta^U,\infty)\\
&\text{(iv)} \quad [\sigma\delta^U,2\sigma\delta^L),[2\sigma\delta^L,2\sigma\delta^U),[2\sigma\delta^U,\infty).
\end{aligned}$$

Since the analysis will not differ greatly between each of these scenarios, we only show the analysis for scenario (ii).

We begin our analysis by letting $\sigma\delta^L < C < 2\sigma\delta^U$. In this case, nature can perturb the demand of one item to either extremal value, but the budget is not yet large enough to consider each item independently. Let

$$s^* = \mu + \frac{b\sigma\delta^U - h\left(C - \sigma\delta^L\right)}{b+h}.$$

For the level of budget in this interval, the six extreme points available to nature are:

$$(\varepsilon_1,\varepsilon_2) \in \left\{ \begin{array}{ccc} \left(-\delta^L, \frac{-\left(C-\sigma\delta^L\right)}{\sigma}\right), & \left(-\delta^L, \delta^U\right), & \left(\frac{-\left(C-\sigma\delta^L\right)}{\sigma}, -\delta^L\right), \\ \left(\delta^U, -\delta^L\right), & \left(\delta^U, \frac{C-\sigma\delta^U}{\sigma}\right), & \left(\frac{C-\sigma\delta^U}{\sigma}, \delta^U\right) \end{array} \right\}.$$

We demonstrate that for $s = s^*$, nature is indifferent among the following actions

$$(\varepsilon_1,\varepsilon_2) \in \left\{ \left(-\delta^L, \frac{-\left(C-\sigma\delta^L\right)}{\sigma}\right), \left(-\delta^L, \delta^U\right), \left(\frac{-\left(C-\sigma\delta^L\right)}{\sigma}, -\delta^L\right), \left(\delta^U, -\delta^L\right) \right\}$$

and that the remaining two actions are suboptimal. That is,

$$\begin{aligned} & g\left(s^*, \left(\frac{-\left(C-\sigma\delta^L\right)}{\sigma}, -\delta^L\right)\right) = g\left(s^*, \left(-\delta^L, \frac{-\left(C-\sigma\delta^L\right)}{\sigma}\right)\right) \\ & = h(s^* - \mu + \sigma\delta^L) + h(s^* - \mu + \left(C-\sigma\delta^L\right)) \\ & = h\left(\frac{b\sigma\delta^U - h\left(C-\sigma\delta^L\right)}{b+h} + \sigma\delta^L\right) + h\left(\frac{b\sigma\delta^U - h\left(C-\sigma\delta^L\right)}{b+h} + C - \sigma\delta^U\right) \\ & = 2h\left(\frac{b\sigma\delta^U - h\left(C-\sigma\delta^L\right)}{b+h}\right) + hC, \text{ and} \end{aligned}$$

$$\begin{aligned} & g\left(s^*, \left(-\delta^L, \delta^U\right)\right) = g\left(s^*, \left(\delta^U, -\delta^L\right)\right) \\ & = b(\mu + \sigma\delta^U - s^*) + h(s^* - \mu + \sigma\delta^L) \\ & = b\left(\sigma\delta^U - \frac{b\sigma\delta^U - h\left(C-\sigma\delta^L\right)}{b+h}\right) + h\left(\frac{b\sigma\delta^U - h\left(C-\sigma\delta^L\right)}{b+h} + \sigma\delta^L\right) \\ & = b\left(\frac{h\sigma\delta^U + h\left(C-\sigma\delta^L\right)}{b+h}\right) + h\left(\frac{b\sigma\delta^U - h\left(C-\sigma\delta^L\right)}{b+h} + \sigma\delta^L\right) \\ & = h\left(\frac{b\sigma\delta^U + b\left(C-\sigma\delta^L\right)}{b+h}\right) + h\left(\frac{b\sigma\delta^U - h\left(C-\sigma\delta^L\right)}{b+h} + \sigma\delta^L\right) \\ & = h\left(\frac{b\sigma\delta^U + b\left(C-\sigma\delta^L\right)}{b+h} + \left(C-\sigma\delta^L\right) - \left(C-\sigma\delta^L\right)\right) + h\left(\frac{b\sigma\delta^U - h\left(C-\sigma\delta^L\right)}{b+h} + \sigma\delta^L\right) \\ & = 2h\left(\frac{b\sigma\delta^U - h\left(C-\sigma\delta^L\right)}{b+h}\right) + hC. \end{aligned}$$

However, we must show that the two remaining extreme points $\left(\delta^U, \frac{C-\sigma\delta^U}{\sigma}\right), \left(\frac{C-\sigma\delta^U}{\sigma}, \delta^U\right)$ are suboptimal.

Note that the above analysis shows that $b(\mu + \sigma\delta^U - s^*) = h(s^* - \mu + C - \sigma\delta^L)$. Hence,

$$g\left(s^*, \left(\frac{C - \sigma\delta^U}{\sigma}, \delta^U\right)\right) = g\left(s^*, \left(\delta^U, \frac{C - \sigma\delta^U}{\sigma}\right)\right) < g\left(s^*, \left(\delta^U, -\delta^L\right)\right),$$

and we see that the remaining two extreme points are suboptimal.

Next, suppose that s^* were not optimal. However, increasing s leads to the following cost:

$$\begin{aligned} g\left(s > s^*, \left(-\delta^L, \frac{-\left(C - \sigma\delta^L\right)}{\sigma}\right)\right) &= 2h(s - \mu) + hC \\ &> 2h(s^* - \mu) + hC \\ &= f(s^*). \end{aligned}$$

Likewise, decreasing s leads to the following cost:

$$\begin{aligned} g\left(s < s^*, \left(\delta^U, -\delta^L\right)\right) &= b(\mu + \sigma\delta^U - s) + h(s - \mu + \sigma\delta^L) \\ &= b(\mu + \sigma\delta^U - s + s^* - s^*) + h(s + s^* - s^* - \mu + \sigma\delta^L) \\ &= b(\mu + \sigma\delta^U - s^*) + h(s^* - \mu + \sigma\delta^L) + (b - h)(s^* - s) \\ &= f(s^*) + (b - h)(s^* - s) > f(s^*). \end{aligned}$$

Therefore, s^* is optimal.

Proceeding with our analysis, assume that $2\sigma\delta^U \le C < 2\sigma\delta^L$. In this setting, nature is able to consider large demands independently, but is not yet able to do so for small demands. Let

$$s^* = \mu + \frac{b\sigma\delta^U - h\left(C - \sigma\delta^L\right)}{b + h}.$$

For this level of budget, the joint constraint for large demands can no longer be active, we end up with a degenerate case leading to the following *five* extreme points:

$$(\varepsilon_1, \varepsilon_2) \in \left\{\left(-\delta^L, \frac{-\left(C - \sigma\delta^L\right)}{\sigma}\right), \left(-\delta^L, \delta^U\right), \left(\frac{-\left(C - \sigma\delta^L\right)}{\sigma}, -\delta^L\right), \left(\delta^U, -\delta^L\right), \left(\delta^U, \delta^U\right)\right\}.$$

Then, for $s = s^*$, nature is indifferent among the following four actions, with the remaining action suboptimal:

$$(\varepsilon_1, \varepsilon_2) \in \left\{ \left(-\delta^L, \frac{-\left(C - \sigma\delta^L\right)}{\sigma} \right), \left(-\delta^L, \delta^U\right), \left(\frac{-\left(C - \sigma\delta^L\right)}{\sigma}, -\delta^L \right), \left(\delta^U, -\delta^L\right) \right\}.$$

But observe that these are the same extreme points and inventory level as in the previous case. Therefore, the analysis is unchanged and s^* is optimal. However, note that in this setting, due to the larger budget, the joint constraint on ε_i^+ cannot be active, and there is only one remaining extreme point $\left(\delta^U, \delta^U\right)$ which is suboptimal. This can be easily seen since the previous analysis shows that $b(\mu + \sigma\delta^U - s^*) = h(s^* - \mu + C - \sigma\delta^L)$. Hence,

$$g\left(s^*, \left(\delta^U, \delta^U\right)\right) = g\left(s^*, \left(\delta^U, \frac{-\left(C - \sigma\delta^L\right)}{\sigma}\right)\right) < g\left(s^*, \left(\delta^U, -\delta^L\right)\right).$$

Finally, if $C \geq 2\delta^L$, then the joint constraints are non-binding and the problem decomposes by item. Then as in the single item case, we set $b(\mu + \sigma\delta^U - s^*) = h(s^* - \mu + \sigma\delta^L)$ and we obtain:

$$s^* = \mu + \frac{b\delta^U - h\delta^L}{b+h}\sigma$$

with corresponding cost

$$f(s^*) = 2h\frac{b\delta^U - h\delta^L}{b+h}\sigma + h(2\sigma\delta^L).$$

Note that in this scenario, because $\delta^L > \delta^U$, the optimal stocking level may lie below the mean.

Recall that there were four original scenarios related to the asymmetry:

$$\begin{array}{rrl}
\text{(i)} & 2\delta^U \leq \delta^L & , \\
\text{(ii)} & \delta^U \leq \delta^L & , \\
 & 2\delta^U > \delta^L & , \\
\text{(iii)} & 2\delta^L \leq \delta^U & , \text{ and} \\
\text{(iv)} & \delta^L \leq \delta^U & , \\
 & 2\delta^L > \delta^U & .
\end{array}$$

While we only analyzed scenario (ii), the analysis is representative and can be easily applied to any of the above scenarios. We summarize the results for all four scenarios below:

Proposition 4.3 *For the two identical item case under asymmetric uncertainty sets,*

(i) Suppose that $2\sigma\delta^U \leq \delta^L$*. Then*

(a) For $\sigma\delta^L \leq C < 2\sigma\delta^L, s^* = \mu + \frac{b\sigma\delta^U - h\left(C - \sigma\delta^L\right)}{b+h}, f(s^*) = 2h\frac{b\sigma\delta^U - h\left(C - \sigma\delta^L\right)}{b+h} + hC,$

(b) For $2\sigma\delta^L \leq C, s^* = \mu + \frac{b\delta^U - h\delta^L}{b+h}, f(s^*) = 2h\frac{b\delta^U - h\delta^L}{b+h} + h(2\sigma\delta^L).$

(ii) Suppose that $\delta^U \leq \delta^L, 2\delta^U > \delta^L$*. Then*

(a) For $\sigma\delta^L \leq C < 2\delta^U, s^* = \mu + \frac{b\sigma\delta^U - h\left(C - \sigma\delta^L\right)}{b+h}, f(s^*) = 2h\frac{b\sigma\delta^U - h\left(C - \sigma\delta^L\right)}{b+h} + hC,$

(b) For $2\sigma\delta^U \leq C < 2\sigma\delta^L, s^* = \mu + \frac{b\sigma\delta^U - h\left(C - \sigma\delta^L\right)}{b+h}, f(s^*) = 2h\frac{b\sigma\delta^U - h\left(C - \sigma\delta^L\right)}{b+h} + hC,$

(c) For $2\sigma\delta^L \leq C, s^* = \mu + \frac{b\delta^U - h\delta^L}{b+h}, f(s^*) = 2h\frac{b\delta^U - h\delta^L}{b+h} + h(2\sigma\delta^L).$

(iii) Suppose that $2\sigma\delta^L \leq \delta^U$*. Then*

(a) For $\sigma\delta^U \leq C < 2\sigma\delta^U, s^* = \mu + \frac{b\sigma\delta^L - h\left(C - \sigma\delta^U\right)}{b+h}, f(s^*) = 2h\frac{b\sigma\delta^L - h\left(C - \sigma\delta^U\right)}{b+h} + hC,$

(b) For $2\sigma\delta^U \leq C, s^* = \mu + \frac{b\delta^U - h\delta^L}{b+h}, f(s^*) = 2h\frac{b\delta^L - h\delta^L}{b+h} + h(2\sigma\delta^L).$

(iv) Suppose that $\delta^L \leq \delta^U, 2\delta^L > \delta^U$*. Then*

(a) For $\sigma\delta^U \leq C < 2\delta^L, s^* = \mu + \frac{b\sigma\delta^L - h\left(C - \sigma\delta^U\right)}{b+h}, f(s^*) = 2h\frac{b\sigma\delta^L - h\left(C - \sigma\delta^U\right)}{b+h} + hC,$

(b) For $2\sigma\delta^L \leq C < 2\sigma\delta^U, s^* = \mu + \frac{b\sigma\delta^L - h\left(C - \sigma\delta^U\right)}{b+h}, f(s^*) = 2h\frac{b\sigma\delta^L - h\left(C - \sigma\delta^U\right)}{b+h} + hC,$

(c) For $2\sigma\delta^U \leq C, s^* = \mu + \frac{b\delta^U - h\delta^L}{b+h}, f(s^*) = 2h\frac{b\delta^L - h\delta^L}{b+h} + h(2\sigma\delta^L).$

The above proposition reveals that the asymmetric two identical item case is quite similar to the symmetric case. Here, the cost is also non-decreasing in C. For each of the above scenarios, the derivative with respect to C begins at $h\left(1 - \frac{2h}{b+h}\right)$, and then becomes 0 as the joint budget constraints become inactive. Additionally, we see that the inventory levels follow the same trend of first increasing to protect against backorders, then decreasing to protect against large holding costs, and finally converging to a fixed value as the budget becomes large enough for both nature and the seller to consider each item separately.

However, in this setting, because the amount by which nature can vary the demand depends on whether nature has chosen to increase or decrease the demand, the stock levels may not converge to the mean. Observe that in all four scenarios examined in Proposition 4.3, s^* converges to $\mu + \frac{b\delta^U - h\delta^L}{b+h}$ as $C \to \infty$. Then the relation of this stock level to the mean depends on the relation of the quantities $b\delta^U$ and $h\delta^L$. These two values represent the incremental costs nature can add for a choice of large demand and small demand, respectively. That is, if $b\delta^U > h\delta^L$, then nature can potentially add higher incremental costs through large demands than through imposing low demands. Therefore, when the seller chooses a stock level so as to make nature indifferent between high demands and low demands, the resulting inventory level will be higher than the mean. Conversely, if $b\delta^U < h\delta^L$, the seller will choose a stock level that is lower than the mean.

Recall that in the symmetric setting, as the ratio of b/h approached 1, the resulting optimal stock level approached μ. This is due to $\delta^U = \delta^L$ in the symmetric setting. In the asymmetric setting, this is no longer the case and the stock level depends on $b\delta^U/h\delta^L$. Then in the asymmetric setting with a large budget of uncertainty, as $b\delta^U/h\delta^L$ approaches 1, the stock level will converge to the mean. Note that this does not necessarily imply that $b = h$ and $\delta^U = \delta^L$, as setting $\delta^U = \frac{h}{b}\delta^L$ is sufficient.

It should also be noted that in the case where μ is small, then δ^L should be bounded by $\frac{\mu}{\sigma}$, so as to prevent demands from becoming negative.

4.2.3 Non-Identical Items

Observe that the previous analysis is not greatly complicated by non-identical demand parameters. That is, we can easily extend the model to include item dependent demand parameters $\sigma_i, \mu_i, \delta_i^U, \delta_i^L$. Consider the following uncertainty set associated with this setting:

$$U(\boldsymbol{\delta}, \delta_Z) = \left\{ \boldsymbol{\varepsilon} : \begin{array}{c} \sum_{i=1}^2 \varepsilon_i^+ \sigma_i \leq \mathbb{E}\left[\tilde{Z}^+\right] + \delta_Z \sqrt{\mathrm{Var}\left(\tilde{Z}^+\right)} \\ \sum_{i=1}^2 \varepsilon_i^- \sigma_i \leq \left|\mathbb{E}\left[\tilde{Z}^-\right]\right| + \delta_Z \sqrt{\mathrm{Var}\left(\tilde{Z}^-\right)} \\ 0 \leq \varepsilon_i^+ \leq \delta_i^U \ \forall\, i \\ 0 \leq \varepsilon_i^- \leq \delta_i^L \ \forall\, i \\ \varepsilon_i = \varepsilon_i^+ - \varepsilon_i^- \ \forall\, i \\ \varepsilon_i^+, \varepsilon_i^- \geq 0 \ \forall\, i \end{array} \right\}. \tag{4.4}$$

Due to these item dependent parameters and the asymmetric uncertainty set, we have 120 scenarios arising from the ranking of $\sigma_1\delta_1^U, \sigma_1\delta_1^L, \sigma_2\delta_2^U, \sigma_2\delta_2^L$ and their sums. That is, there is a scenario associated with each consistent ordering. Note, however, that certain orderings are not consistent and cannot occur, i.e. situations such as $\sigma_1\delta_1^L + \sigma_2\delta_2^L > \sigma_1\delta_1^L + \sigma_2\delta_2^L + \sigma_1\delta_1^U$.

However, we can again express the general extreme points in closed form independently of the ranking of the uncertainty parameters as we did in (4.2). Note that these strongly resemble the extreme points already listed in the previous section:

(i) $\left(-\left(\frac{C}{\sigma_1} \wedge \delta_1^L\right), -\left[\left(\frac{C-\sigma_1\delta_1^L}{\sigma_2}\right)^+ \wedge \delta_2^L\right]\right)$, (iv) $\left(\left[\left(\frac{C-\sigma_2\delta_2^U}{\sigma_1}\right)^+ \wedge \delta_1^U\right], \left(\frac{C}{\sigma_2} \wedge \delta_2^U\right)\right)$,

(ii) $\left(-\left[\left(\frac{C-\sigma_2\delta_2^L}{\sigma_1}\right)^+ \wedge \delta_1^L\right], -\left(\frac{C}{\sigma_2} \wedge \delta_2^L\right)\right)$, (v) $\left(-\left(\frac{C}{\sigma_1} \wedge \delta_1^L\right), \left(\frac{C}{\sigma_2} \wedge \delta_2^U\right)\right)$, and

(iii) $\left(\left(\frac{C}{\sigma_1} \wedge \delta_1^U\right), \left[\left(\frac{C-\sigma_1\delta^U}{\sigma_2}\right)^+ \wedge \delta_2^U\right]\right)$, (vi) $\left(\left(\frac{C}{\sigma_1} \wedge \delta_1^U\right), -\left(\frac{C}{\sigma_2} \wedge \delta_2^L\right)\right)$.

As before, we begin by illustrating the dependence of the feasible region for $\boldsymbol{\varepsilon}$ on C by fixing a few parameters $\delta_1^U = \delta_1^L = \delta_2^U = \delta_2^L = 2$, $\sigma_1 = 2$, and $\sigma_2 = 1$. The graphs of the feasible region for several different values of C are shown in Figure 8. Observe that when compared to the feasible region depicted in Figure 3, the item dependent demand parameters have resulted in skewing the the hexagonal regions. If the individual bounds δ_i^L, δ_i^U were different, this would simply result in a black rectangle rather than a black square.

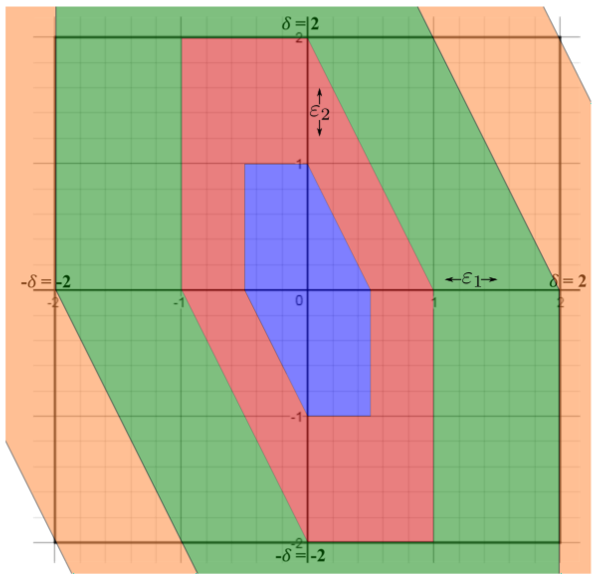

Figure 8: Feasible region for nature under a variety of budgets with $\delta_1^U = \delta_1^L = \delta_2^U = \delta_2^L = 2$, $\sigma_1 = 2$, $\sigma_2 = 1$: a) $C = 1$ in blue, b) $C = 2$ in red, c) $C = 4$ in green, d) $C = 6$ in orange.

We next depict $f(s)$ below in Figure 9.

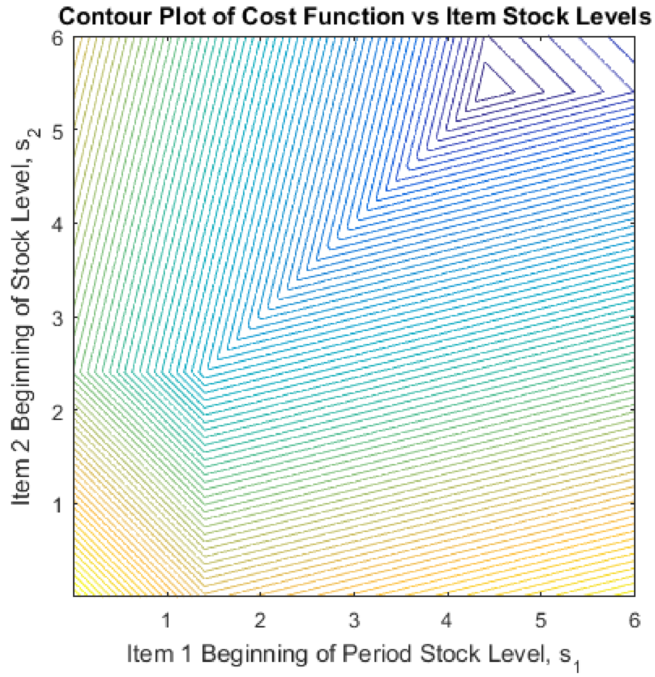

Figure 9: Contour plot for two item case with cost parameters $b_1 = b_2 = 5, h_1 = h_2 = 0.5$, and demand parameters $\mu_1 = 3, \mu_2 = 2, \sigma_1 = 3, \sigma_2 = 2, \delta_1^U = \delta_1^L = 2.5, \delta_2^U = \delta_2^L = 2, C = 3$.

As before, the contour plot seems to split into four regions. Again, this is indicative of 4 different actions chosen by nature to inflict the maximal cost for these inventory levels. Now observe that the minimum in the contour plot seems to occur at the intersection of these regions. We will use this observation subsequently.

We will analyze only one of these scenarios under the assumption that demand for item 1 is more variable than that of item 2. Specifically, we assume that the bounds on deviation for item 1 are larger than those for item 2, which we believe is more common in practice. We additionally assume that positive deviations can potentially exceed negative deviations. That is, nature has more power to choose large demands than to choose small demands. Mathematically, we have that $\sigma_1\delta_1^U > \sigma_2\delta_2^U > \sigma_1\delta_1^L > \sigma_2\delta_2^L$. Additionally, we must also assume an ordering on their partial sums. In this analysis we will assume $\sigma_1\delta_1^U > \sigma_2\delta_2^L + \sigma_1\delta_1^L > \sigma_2\delta_2^U$ and $\sigma_1\delta_1^U < \sigma_2\delta_2^U + \sigma_2\delta_2^L + \sigma_1\delta_1^L$. However, the analysis can be easily extended to any of the possible scenarios.

The above assumptions yield a natural partition of the non-negative real numbers:

(i) $[0, \sigma_2\delta_2^L)$,

(ii) $[\sigma_2\delta_2^L, \sigma_1\delta_1^L)$,

(iii) $[\sigma_1\delta_1^L, \sigma_2\delta_2^U)$,

(iv) $[\sigma_2\delta_2^U, \sigma_1\delta_1^L + \sigma_2\delta_2^L)$,

(v) $[\sigma_1\delta_1^L + \sigma_2\delta_2^L, \sigma_1\delta_1^U)$,

(vi) $[\sigma_1\delta_1^U, \sigma_1\delta_1^U + \sigma_2\delta_2^U)$, and

(vii) $[\sigma_1\delta_1^U + \sigma_2\delta_2^U, \infty)$.

Recall that we had previously assumed $C > \max\left\{\sigma\delta^L, \sigma\delta^U\right\}$ in Section 4.2.3 when items had identical demand parameters. In this non-identical item setting, we will generalize the assumption to $C > \max\left\{\max_i \sigma_i\delta_i^L, \max_i \sigma_i\delta_i^U\right\}$. Therefore, we are interested only in intervals (vi) and (vii) stated above.

Beginning with case (vi), nature can choose minimal or maximal demands for either item. However, while nature can choose a minimal demand for both items simultaneously, nature cannot simultaneously choose maximal demands for both items. Note that the joint constraint on small demands is then inactive, resulting in a degeneracy. Hence, there are only the following five extreme points for nature:

$$(\varepsilon_1, \varepsilon_2) \in \left\{\left(-\delta_1^L, -\delta_2^L\right), \left(-\delta_1^L, \delta_2^U\right), \left(\delta_1^U, -\delta_2^L\right), \left(\frac{C - \sigma_2\delta_2^U}{\sigma_1}, \delta_2^U\right), \left(\delta_1^U, \frac{C - \sigma_1\delta_1^U}{\sigma_2}\right)\right\}.$$

Now, let

$$\begin{aligned} s_1^* &= \mu_1 + \frac{b\delta_1^U - h\delta_1^L}{b+h}\sigma_1, \\ s_2^* &= \mu_2 + \frac{b\delta_2^U - h\delta_2^L}{b+h}\sigma_2, \end{aligned} \tag{4.5}$$

and observe that at $s_i = s_i^* \ \forall\ i$, nature is indifferent among the following set of extreme points:

$$(\varepsilon_1, \varepsilon_2) \in \left\{\left(-\delta_1^L, -\delta_2^L\right), \left(-\delta_1^L, \delta_2^U\right), \left(\delta_1^U, -\delta_2^L\right)\right\}.$$

As noted earlier, because of our assumption that $C > \sigma_1\delta_1^U$ and $\sigma_1\delta_1^U > \sigma_2\delta_2^L + \sigma_1\delta_1^L$, the joint constraint on low demands will not be active, and as a result, we have three extreme points. Now observe from (4.5) that $s_i^* > \mu_i \ \forall\ i$ as we have assumed $b > h, \delta_i^U \geq \delta_i^L$. Then for the first extreme point, nature associates holding costs for both items to inflict a cost of

$$g\left(\boldsymbol{s}^{*},\left(-\delta_1^L,-\delta_2^L\right)\right)=h\sum_{i=1}^{2}(s_i^*-\mu_i)+h(\sigma_1\delta_1^L+\sigma_2\delta_2^L).$$

Now consider the cost under the remaining two extreme points which associate a holding cost with one item and a backorder cost with the second item, and observe that

$$\begin{aligned}
g\left(\boldsymbol{s}^{*},\left(-\delta_1^L,\delta_2^U\right)\right)&=h(s_1^*-\mu_1+\sigma_1\delta_1^L)+b(\mu_2+\sigma_2\delta_2^U-s_2^*)\\
&=h(s_1^*-\mu_1+\sigma_1\delta_1^L)+b\left(\sigma_2\delta_2^U-\frac{b\sigma_2\delta_2^U-h\sigma_2\delta_2^L}{b+h}\right)\\
&=h(s_1^*-\mu_1+\sigma_1\delta_1^L)+b\left(\frac{h\sigma_2\delta_2^U+h\sigma_2\delta_2^L}{b+h}\right)\\
&=h(s_1^*-\mu_1+\sigma_1\delta_1^L)+h\left(\frac{b\sigma_2\delta_2^U+b\sigma_2\delta_2^L}{b+h}\right)\\
&=h(s_1^*-\mu_1+\sigma_1\delta_1^L)+h\left(\frac{b\sigma_2\delta_2^U-h\sigma_2\delta_2^L+(b+h)\sigma_2\delta_2^L}{b+h}\right)\\
&=h(s_1^*-\mu_1)+h\left(\frac{b\sigma_2\delta_2^U-h\sigma_2\delta_2^L}{b+h}\right)+h(\sigma_1\delta_1^L+\sigma_2\delta_2^L)\\
&=h(s_1^*-\mu_1)+h\left(\mu_2+\frac{b\sigma_2\delta_2^U-h\sigma_2\delta_2^L}{b+h}-\mu_2\right)+h(\sigma_1\delta_1^U+\sigma_2\delta_2^U)\\
&=h(s_1^*-\mu_1)+h\left(s_2^*-\mu_2\right)+h(\sigma_1\delta_1^L+\sigma_2\delta_2^L),
\end{aligned}$$

and

$$\begin{aligned}
g\left(\boldsymbol{s}^{*},\left(\delta_1^U,-\delta_2^L\right)\right)&=b(\mu_1+\sigma_1\delta_1^U-s_1^*)+h(s_2^*-\mu_2+\sigma_2\delta_2^L)\\
&=b\left(\sigma_1\delta_1^U-\frac{b\sigma_1\delta_1^U-h\sigma_1\delta_1^L}{b+h}\right)+h(s_2^*-\mu_2+\sigma_2\delta_2^L)\\
&=b\left(\frac{h\sigma_1\delta_1^U+h\sigma_1\delta_1^L}{b+h}\right)+h(s_2^*-\mu_2+\sigma_2\delta_2^L)\\
&=h\left(\frac{b\sigma_1\delta_1^U+b\sigma_1\delta_1^L}{b+h}\right)+h(s_2^*-\mu_2+\sigma_2\delta_2^L)\\
&=h\left(\frac{b\sigma_1\delta_1^U-h\sigma_1\delta_1^L+(b+h)\sigma_1\delta_1^L}{b+h}\right)+h(s_2^*-\mu_2+\sigma_2\delta_2^L)\\
&=h\left(\frac{b\sigma_1\delta_1^U-h\sigma_1\delta_1^L}{b+h}\right)+h(s_2^*-\mu_2)+h(\sigma_1\delta_1^L+\sigma_2\delta_2^L)\\
&=h\left(\mu_1+\frac{b\sigma_1\delta_1^U-h\sigma_1\delta_1^L}{b+h}-\mu_1\right)+h(s_2^*-\mu_2)+h(\sigma_1\delta_1^L+\sigma_2\delta_2^L)\\
&=h\left(s_1^*-\mu_1\right)+h(s_2^*-\mu_2)+h(\sigma_1\delta_1^L+\sigma_2\delta_2^L).
\end{aligned}$$

Therefore, these actions yield the same cost. The remaining two extreme points to consider are $\left(\frac{C-\sigma_2\delta_2^U}{\sigma_1}, \delta_2^U\right)$ and $\left(\delta_1^U, \frac{C-\sigma_1\delta_1^U}{\sigma_2}\right)$ which associate backorder costs with both items. However, these are suboptimal for nature as we now demonstrate. Observe that our analysis showed that at $\boldsymbol{s}^*$,

$$\begin{aligned} b(\mu_1 + (C - \sigma_2\delta_2^U) - s_1^*) &< b(\mu_1 + \sigma_1\delta_1^U - s_1^*) \\ &= h\left(s_1^* - \mu_1 + \sigma_1\delta_1^L\right), \end{aligned}$$

which implies that $g\left(\boldsymbol{s}^*, \left(-\delta_1^L, \delta_2^U\right)\right) > g\left(\boldsymbol{s}^*, \left(\frac{C-\sigma_2\delta_2^U}{\sigma_1}, \delta_2^U\right)\right)$. Observe also that at $\boldsymbol{s}^*$,

$$\begin{aligned} b(\mu_2 + (C - \sigma_1\delta_1^U) - s_2^*) &< b(\mu_2 + \sigma_2\delta_2^U - s_2^*) \\ &= h\left(s_2^* - \mu_2 + \sigma_2\delta_2^L\right), \end{aligned}$$

which implies that $g\left(\boldsymbol{s}^*, \left(\delta_1^L, -\delta_2^U\right)\right) > g\left(\boldsymbol{s}^*, \left(\delta_1^U, \frac{C-\sigma_1\delta_1^U}{\sigma_2}\right)\right)$. Therefore, these two points are suboptimal and $f(\boldsymbol{s}^*) = h\sum_i (s_i^* - \mu_i) + hC$.

Now suppose that $\boldsymbol{s}^*$ were not optimal. First, note that the seller cannot increase the stock level of either item as

$$g\left(\boldsymbol{s} \geq \boldsymbol{s}^*, \left(-\delta_1^L, -\delta_2^L\right)\right) = f(\boldsymbol{s}^*) + h\sum_{i=1}^{2}(s_i - s_i^*).$$

Next suppose that the seller attempted to decrease the stock levels of both items. That is, $s_1 = s_1^* - \eta_1, s_2 = s_2^* - \eta_2$ with $\eta_1, \eta_2 \geq 0$. But suppose that $\eta_1 > \eta_2$ and note that

$$\begin{aligned} g\left(\boldsymbol{s}, \left(\delta_1^U, -\delta_2^L\right)\right) &= b(\mu_1 + \sigma_1\delta_1^U - (s_1^* - \eta_1)) + h((s_2^* - \eta_2) - \mu_2 + \sigma_2\delta_2^L) \\ &= b(\mu_1 + \sigma_1\delta_1^U - s_1^*) + h(s_2^* - \mu_2 + \sigma_2\delta_2^L) + b\eta_1 - h\eta_2 \\ &= f(s^*) + b\eta_1 - h\eta_2 > f(s^*). \end{aligned}$$

A similar argument shows that the cost to the seller must also increase for $\eta_2 > \eta_1$.

Now suppose instead that the seller attempted to increase the stock level of the first item, while decreasing the stock level of the second item. Let $s_1 = s_1^* - \eta_1, s_2 = s_2^* + \eta_2$ with

$\eta_1, \eta_2 \geq 0$ and observe that

$$\begin{aligned} g\left(\boldsymbol{s}, \left(\delta_1^U, -\delta_2^L\right)\right) &= b(\mu_1 + \sigma_1\delta_1^U - (s_1^* - \eta_1)) + h((s_2^* + \eta_2) - \mu_2 + \sigma_2\delta_2^L) \\ &= b(\mu_1 + \sigma_1\delta_1^U - s_1^*) + h(s_2^* - \mu_2 + \sigma_2\delta_2^L) + b\eta_1 + h\eta_2 \\ &= f(s^*) + b\eta_1 + h\eta_2 > f(s^*). \end{aligned}$$

A similar argument shows that the cost to the seller must also increase for $s_1 = s_1^* + \eta_1, s_2 = s_2^* - \eta_2$ with $\eta_1, \eta_2 \geq 0$. Therefore, s^* is optimal for $C \in [\sigma_1\delta_1^U, \sigma_1\delta_1^U + \sigma_2\delta_2^U)$.

Finally, we consider interval (vii). Recall that for interval (vii), $C > \sigma_1\delta_1^U + \sigma_2\delta_2^U$. Note that in this interval C is sufficiently large that both nature and the seller will consider each item independently. That is, the problem separates and the solution

$$\begin{aligned} s_1^* &= \mu_1 + \frac{b\delta_1^U - h\delta_1^L}{b+h}\sigma_1, \\ s_2^* &= \mu_2 + \frac{b\delta_2^U - h\delta_2^L}{b+h}\sigma_2, \end{aligned}$$

is optimal for the seller.

We summarize the above analysis and results with the following proposition.

Proposition 4.4 *For the asymmetric two non-identical item case, under assumptions*

(i) $C > \max\left\{\max_i \sigma_i\delta_i^L, \max_i \sigma_i\delta_i^U\right\}$,

(ii) $\sigma_1\delta_1^U > \sigma_2\delta_2^U > \sigma_1\delta_1^L > \sigma_2\delta_2^L$,

(iii) $\sigma_1\delta_1^U > \sigma_2\delta_2^L + \sigma_1\delta_1^L > \sigma_2\delta_2^U$, *and*

(iv) $\sigma_1\delta_1^U < \sigma_2\delta_2^U + \sigma_2\delta_2^L + \sigma_1\delta_1^L$,

the optimal set of stock levels and corresponding cost are:

$$\begin{aligned} s_i^* &= \mu_i + \frac{b\delta_i^U - h\delta_i^L}{b+h}\sigma_i \\ f(s^*) &= h\sum_{i=1}^{2} \frac{b\delta_i^U - h\delta_i^L}{b+h}\sigma_i + h(\sigma_1\delta_1^L + \sigma_2\delta_2^L). \end{aligned}$$

Note that this strongly resembles the previous asymmetric two identical item case summarized in Proposition 4.3. However, note that the fact that costs were identical among items was not used in our analysis. Recall that the analysis utilizes the extreme points of nature's feasible region and finds a set of stock levels that make nature indifferent between assigning holding costs and backorder costs to each item. This analysis can easily be extended to the case of item dependent costs. In the following section, we relax the assumption that each item has identical holding and backorder costs.

4.2.4 Non-Identical Cost Case

We now consider the case where holding and backorder costs may differ by item. First, note that the item holding and backorder costs do not appear in the uncertainty set, and therefore, the uncertainty set will be exactly the same as in the previous section. That is,

$$U(\boldsymbol{\delta}, \delta_Z) = \left\{ \boldsymbol{\varepsilon} : \begin{array}{c} \sum_{i=1}^{2} \varepsilon_i^+ \sigma_i \leq \mathbb{E}\left[\tilde{Z}^+\right] + \delta_Z \sqrt{\mathrm{Var}\left(\tilde{Z}^+\right)} \\ \sum_{i=1}^{2} \varepsilon_i^- \sigma_i \leq \left|\mathbb{E}\left[\tilde{Z}^-\right]\right| + \delta_Z \sqrt{\mathrm{Var}\left(\tilde{Z}^-\right)} \\ 0 \leq \varepsilon_i^+ \leq \delta_i^U \ \forall\, i \\ 0 \leq \varepsilon_i^- \leq \delta_i^L \ \forall\, i \\ \varepsilon_i = \varepsilon_i^+ - \varepsilon_i^- \ \forall\, i \\ \varepsilon_i^+, \varepsilon_i^- \geq 0 \ \forall\, i \end{array} \right\}.$$

Consequently, the extreme points available to nature are the same as in the previous section. However, the seller now has the following mathematical program to solve

$$\begin{aligned} \min_s \max_\varepsilon \sum_{i=1}^{2} \quad & \max\left\{b_i(d_i - s_i), h_i(s_i - d_i)\right\} \\ \text{subject to} \quad & \varepsilon \in U(\boldsymbol{\delta}, \delta_Z). \end{aligned}$$

Recall that we previously assumed that $b > h$ when each item had identical backorder and holding costs. In this setting, we will assume that $\min_i b_i > \max_i h_i$. Because the feasible region for nature has not changed, we still have the same 120 scenarios arising from the ranking of $\sigma_1\delta_1^U, \sigma_1\delta_1^L, \sigma_2\delta_2^U, \sigma_2\delta_2^L$ and their sums.

In this section, we will analyze the same scenario as before, under the assumption that demand for item 1 is more variable than that of item 2. Specifically, we make the same assumption that both bounds on deviation for item 1 are larger than those of item 2. We also assume that the bounds on positive deviation are larger than those of negative deviation. Mathematically, we are assuming that $\sigma_1\delta_1^U > \sigma_2\delta_2^U > \sigma_1\delta_1^L > \sigma_2\delta_2^L$. Additionally, we also assume that $\sigma_1\delta_1^U > \sigma_2\delta_2^L + \sigma_1\delta_1^L > \sigma_2\delta_2^U$ and $\sigma_1\delta_1^U < \sigma_2\delta_2^U + \sigma_2\delta_2^L + \sigma_1\delta_1^L$. Again, the analysis can be easily modified to apply to any of the possible scenarios.

The above assumptions yield the same partition of the non-negative real numbers as seen in the previous section:

(i) $[0, \sigma_2\delta_2^L)$,

(ii) $[\sigma_2\delta_2^L, \sigma_1\delta_1^L)$,

(iii) $[\sigma_1\delta_1^L, \sigma_2\delta_2^U)$,

(iv) $[\sigma_2\delta_2^U, \sigma_1\delta_1^L + \sigma_2\delta_2^L)$,

(v) $[\sigma_1\delta_1^L + \sigma_2\delta_2^L, \sigma_1\delta_1^U)$,

(vi) $[\sigma_1\delta_1^U, \sigma_1\delta_1^U + \sigma_2\delta_2^U)$, and

(vii) $[\sigma_1\delta_1^U + \sigma_2\delta_2^U, \infty)$.

However, under our assumption that $C > \max\left\{\max_i \sigma_i\delta_i^L, \max_i \sigma_i\delta_i^U\right\}$, we restrict our attention to intervals (vi) and (vii).

Beginning with case (vi), the extreme points for nature are again

$$(\varepsilon_1, \varepsilon_2) \in \left\{\left(-\delta_1^L, -\delta_2^L\right), \left(-\delta_1^L, \delta_2^U\right), \left(\delta_1^U, -\delta_2^L\right), \left(\frac{C - \sigma_2\delta_2^U}{\sigma_1}, \delta_2^U\right), \left(\delta_1^U, \frac{C - \sigma_1\delta_1^U}{\sigma_2}\right)\right\}.$$

Now, let

$$s_1^* = \mu_1 + \frac{b_1\delta_1^U - h_1\delta_1^L}{b_1 + h_1}\sigma_1,$$
$$s_2^* = \mu_2 + \frac{b_2\delta_2^U - h_2\delta_2^L}{b_2 + h_2}\sigma_2,$$

and observe that at $s_i = s_i^* \ \forall\ i$, nature is indifferent among the following set of extreme points:

$$(\varepsilon_1, \varepsilon_2) \in \left\{ \left(-\delta_1^L, -\delta_2^L\right), \left(-\delta_1^L, \delta_2^U\right), \left(\delta_1^U, -\delta_2^L\right) \right\}.$$

Note that as before, because the joint constraint on low demands is no longer binding, we only have three extreme points. First, we consider extreme point $\left(-\delta_1^L, -\delta_2^L\right)$. Observe that because $s_i^* > \mu_i \ \forall \ i$, choosing minimal demands for both items inflicts cost

$$g\left(\boldsymbol{s^*}, \left(-\delta_1^L, -\delta_2^L\right)\right) = \sum_{i=1}^{2} h_i(s_i^* - \mu_i) + (h_1\sigma_1\delta_1^L + h_2\sigma_2\delta_2^L).$$

Now consider the cost under the remaining two extreme points $\left(-\delta_1^L, \delta_2^U\right)$ and $\left(\delta_1^U, -\delta_2^L\right)$, and observe that

$$\begin{aligned}
g\left(\boldsymbol{s^*}, \left(-\delta_1^L, \delta_2^U\right)\right) &= h_1(s_1^* - \mu_1 + \sigma_1\delta_1^L) + b_2(\mu_2 + \sigma_2\delta_2^U - s_2^*) \\
&= h_1(s_1^* - \mu_1 + \sigma_1\delta_1^L) + b_2\left(\sigma_2\delta_2^U - \frac{b_2\sigma_2\delta_2^U - h_2\sigma_2\delta_2^L}{b_2 + h_2}\right) \\
&= h_1(s_1^* - \mu_1 + \sigma_1\delta_1^L) + b_2\left(\frac{h_2\sigma_2\delta_2^U + h_2\sigma_2\delta_2^L}{b_2 + h_2}\right) \\
&= h_1(s_1^* - \mu_1 + \sigma_1\delta_1^L) + h_2\left(\frac{b_2\sigma_2\delta_2^U + b_2\sigma_2\delta_2^L}{b_2 + h_2}\right) \\
&= h_1(s_1^* - \mu_1 + \sigma_1\delta_1^L) + h_2\left(\frac{b_2\sigma_2\delta_2^U - h_2\sigma_2\delta_2^L + (b_2 + h_2)\sigma_2\delta_2^L}{b_2 + h_2}\right) \\
&= h_1(s_1^* - \mu_1) + h_2\left(\frac{b_2\sigma_2\delta_2^U - h_2\sigma_2\delta_2^L}{b_2 + h_2}\right) + h_1\sigma_1\delta_1^L + h_2\sigma_2\delta_2^L \\
&= h_1(s_1^* - \mu_1) + h_2\left(\mu_2 + \frac{b_2\sigma_2\delta_2^U - h_2\sigma_2\delta_2^L}{b_2 + h_2} - \mu_2\right) + h_1\sigma_1\delta_1^L + h_2\sigma_2\delta_2^L \\
&= h_1(s_1^* - \mu_1) + h_2\left(s_2^* - \mu_2\right) + h_1\sigma_1\delta_1^L + h_2\sigma_2\delta_2^L
\end{aligned}$$

and

$$
\begin{aligned}
g\left(\boldsymbol{s}^*, \left(\delta_1^U, -\delta_2^L\right)\right) &= b_1(\mu_1 + \sigma_1\delta_1^U - s_1^*) + h_2(s_2^* - \mu_2 + \sigma_2\delta_2^L) \\
&= b_1\left(\sigma_1\delta_1^U - \frac{b\sigma_1\delta_1^U - h\sigma_1\delta_1^L}{b_1+h_1}\right) + h_2(s_2^* - \mu_2 + \sigma_2\delta_2^L) \\
&= b_1\left(\frac{h_1\sigma_1\delta_1^U + h_1\sigma_1\delta_1^L}{b_1+h_1}\right) + h_2(s_2^* - \mu_2 + \sigma_2\delta_2^L) \\
&= h_1\left(\frac{b_1\sigma_1\delta_1^U + b_1\sigma_1\delta_1^L}{b_1+h_1}\right) + h_2(s_2^* - \mu_2 + \sigma_2\delta_2^L) \\
&= h_1\left(\frac{b_1\sigma_1\delta_1^U - h_1\sigma_1\delta_1^L + (b_1+h_1)\sigma_1\delta_1^L}{b_1+h_1}\right) + h_2(s_2^* - \mu_2 + \sigma_2\delta_2^L) \\
&= h_1\left(\frac{b_1\sigma_1\delta_1^U - h_1\sigma_1\delta_1^L}{b_1+h_1}\right) + h_2(s_2^* - \mu_2) + h_1\sigma_1\delta_1^L + h_2\sigma_2\delta_2^L \\
&= h_1\left(\mu_1 + \frac{b\sigma_1\delta_1^U - h_1\sigma_1\delta_1^L}{b_1+h_1} - \mu_1\right) + h_2(s_2^* - \mu_2) + h_1\sigma_1\delta_1^L + h_2\sigma_2\delta_2^L) \\
&= h_1\left(s_1^* - \mu_1\right) + h_2(s_2^* - \mu_2) + h_1\sigma_1\delta_1^L + h_2\sigma_2\delta_2^L).
\end{aligned}
$$

Therefore, these actions yield the same cost. The remaining two extreme points to consider are $\left(\frac{C-\sigma_2\delta_2^U}{\sigma_1}, \delta_2^U\right)$ and $\left(\delta_1^U, \frac{C-\sigma_1\delta_1^U}{\sigma_2}\right)$. However, these are suboptimal for nature, as we now show.

Observe that the above analysis showed that at $\boldsymbol{s}^*$,

$$
\begin{aligned}
b_1(\mu_1 + (C - \sigma_2\delta_2^U) - s_1^*) &< b_1(\mu_1 + \sigma_1\delta_1^U - s_1^*) \\
&= h_1\left(s_1^* - \mu_1 + \sigma_1\delta_1^L\right),
\end{aligned}
$$

which implies that $g\left(\boldsymbol{s}^*, \left(-\delta_1^L, \delta_2^U\right)\right) > g\left(\boldsymbol{s}^*, \left(\frac{C-\sigma_2\delta_2^U}{\sigma_1}, \delta_2^U\right)\right)$. Observe also that at $\boldsymbol{s}^*$,

$$
\begin{aligned}
b_2(\mu_2 + (C - \sigma_1\delta_1^U) - s_2^*) &< b_2(\mu_2 + \sigma_2\delta_2^U - s_2^*) \\
&= h_2\left(s_2^* - \mu_2 + \sigma_2\delta_2^L\right),
\end{aligned}
$$

which implies that $g\left(\boldsymbol{s}^*, \left(\delta_1^L, -\delta_2^U\right)\right) > g\left(\boldsymbol{s}^*, \left(\delta_1^U, \frac{C-\sigma_1\delta_1^U}{\sigma_2}\right)\right)$. Therefore, these two points are suboptimal and $f(\boldsymbol{s}^*) = \sum_i h_i(s_i^* - \mu_i) + h_1\sigma_1\delta_1^L + h_2\sigma_2\delta_2^L$.

Now suppose that $\boldsymbol{s}^*$ were not optimal. First, note that the seller cannot increase the stock level of either item as

$$g\left(\boldsymbol{s} \geq \boldsymbol{s}^{*}, \left(-\delta_1^L, -\delta_2^L\right)\right) = f(\boldsymbol{s}^{*}) + \sum_{i=1}^{2} h_i(s_i - s_i^*).$$

Now suppose instead that the seller attempted to increase the stock level of the first item, while decreasing the stock level of the second item. Let $s_1 = s_1^* - \eta_1, s_2 = s_2^* + \eta_2$ with $\eta_1, \eta_2 \geq 0$ and observe that

$$\begin{aligned}
g\left(\boldsymbol{s}, \left(\delta_1^U, -\delta_2^L\right)\right) &= b_1(\mu_1 + \sigma_1\delta_1^U - (s_1^* - \eta_1)) + h_2((s_2^* + \eta_2) - \mu_2 + \sigma_2\delta_2^L) \\
&= b_1(\mu_1 + \sigma_1\delta_1^U - s_1^*) + h_2(s_2^* - \mu_2 + \sigma_2\delta_2^L) + b_1\eta_1 + h_2\eta_2 \\
&= f(s^*) + b_1\eta_1 + h_2\eta_2 \\
&\geq f(s^*).
\end{aligned}$$

A similar argument shows that the cost to the seller must also increase for $s_1 = s_1^* + \eta_1, s_2 = s_2^* - \eta_2$ with $\eta_1, \eta_2 \geq 0$.

Next, suppose that the seller attempted to decrease the stock levels of both items. That is, $s_1 = s_1^* - \eta_1, s_2 = s_2^* - \eta_2$ with $\eta_1, \eta_2 \geq 0$. Note that

$$\begin{aligned}
g\left(\boldsymbol{s}, \left(\delta_1^U, -\delta_2^L\right)\right) &= b_1(\mu_1 + \sigma_1\delta_1^U - (s_1^* - \eta_1)) + h_2((s_2^* - \eta_2) - \mu_2 + \sigma_2\delta_2^L) \\
&= b_1(\mu_1 + \sigma_1\delta_1^U - s_1^*) + h_2(s_2^* - \mu_2 + \sigma_2\delta_2^L) + b_1\eta_1 - h_2\eta_2 \qquad (4.6) \\
&= f(s^*) + b_1\eta_1 - h_2\eta_2
\end{aligned}$$

and

$$\begin{aligned}
g\left(\boldsymbol{s}, \left(-\delta_1^L, \delta_2^U\right)\right) &= h_1(s_1^* - \eta_1 - \mu_1 + \sigma_1\delta_1^L) + b_2(\mu_2 + \sigma_2\delta_2^U - (s_2^* - \eta_2)) \\
&= h_1(s_1^* - \mu_1 + \sigma_1\delta_1^L) + b_2(\mu_2 + \sigma_2\delta_2^U - s_2^* - \eta_2) + b_2\eta_2 - h_1\eta_1 \qquad (4.7) \\
&= f(s^*) + b_2\eta_2 - h_1\eta_1.
\end{aligned}$$

Then if $\eta_1 > \eta_2$, nature can select action $\left(\delta_1^U, -\delta_2^L\right)$ to inflict higher costs, and if $\eta_1 < \eta_2$, nature can select action $\left(-\delta_1^L, \delta_2^U\right)$ to inflict higher costs, and therefore, s^* is optimal for $C \in [\sigma_1\delta_1^U, \sigma_1\delta_1^U + \sigma_2\delta_2^U)$.

Finally, we consider interval (vii). But note that in this interval C is sufficiently large so that both nature and the seller will consider each item independently. That is, the problem separates into independent single-item subproblems and

$$s_1^* = \mu_1 + \frac{b_1\delta_1^U - h_1\delta_1^L}{b_1 + h_1}\sigma_1,$$
$$s_2^* = \mu_2 + \frac{b_2\delta_2^U - h_2\delta_2^L}{b_2 + h_2}\sigma_2,$$

is optimal for the seller.

We summarize the above analysis and results with the following proposition.

Proposition 4.5 *For the asymmetric two non-identical item case with non-identical cost parameters, under assumptions*

1. $C > \max\left\{\max_i \sigma_i\delta_i^L, \max_i \sigma_i\delta_i^U\right\}$,

2. $\sigma_1\delta_1^U > \sigma_2\delta_2^U > \sigma_1\delta_1^L > \sigma_2\delta_2^L$,

3. $\sigma_1\delta_1^U > \sigma_2\delta_2^L + \sigma_1\delta_1^L > \sigma_2\delta_2^U$, *and*

4. $\sigma_1\delta_1^U < \sigma_2\delta_2^U + \sigma_2\delta_2^L + \sigma_1\delta_1^L$,

(i) If $\sigma_1\delta_1^U \leq C < \sigma_1\delta_1^U + \sigma_2\delta_2^U$, *then* $s_i^* = \mu_i + \frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i$ *is optimal with cost* $f(s^*) = \sum_{i=1}^2 h_i\left[\frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i + \sigma_i\delta_i^L\right]$.

(ii) If $\sigma_1\delta_1^U + \sigma_2\delta_2^U \leq C$, *then* $s_i^* = \mu_i + \frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i$ *is optimal with cost* $f(s^*) = \sum_{i=1}^2 h_i\left[\frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i + \sigma_i\delta_i^L\right]$.

Next, we extend our analysis to the case where there are $n > 2$ items to consider, and further relax the problem by allowing for non-identical budgets of uncertainty.

4.3 Multiple Items

While we were able to characterize the asymmetric non-identical two item case under non-identical cost parameters and obtain closed form solutions, closed form solutions do not exist for the general case. Specifically, every analysis in the previous section, section 4.2, relied on the fact that $\min_i b_i > \max_i h_i$. However, this condition is not sufficient for multiple items. For a case with n items, the analogous condition is

$$\frac{\min_i b_i}{\max_i h_i} > n - 1.$$

This can be a reasonable assumption for relatively small n, but is not likely to be satisfied for large n. First, suppose that this condition holds. Below, we demonstrate how to extend our previous analysis under this assumption. Recall that the extreme points considered previously have a common form. That is, for each item i, the stock level was chosen so that nature is indifferent between two types of extreme points:

(1) $\hat{\boldsymbol{\varepsilon}}$, an extreme point which chooses small demands for all items.

(2) $\tilde{\boldsymbol{\varepsilon}}(i)$, an extreme point which chooses the maximal demand for i while choosing small demands for all other items $j \neq i$.

However, note that because costs are no longer identical, nature has a clear preference for certain items. Now recall that in Section 3.1 we first defined the uncertainty set utilizing two parameters δ_Z^+, δ_Z^- that governed the maximum aggregate positive and negative deviation in demands respectively. Previously, we had assumed that the two parameters were identical and could be represented by a single parameter δ_Z. In this section, we now relax that assumption and consider the fully generalized parameter set which allows for non-identical budgets of uncertainty, non-identical demand parameters, and non-identical cost parameters. That is, denote the positive and negative budgets of uncertainty to be C^+ and C^-, respectively, where C^+ may differ from C^-. We further assume that $C^+ \geq C^-$. This stems from our assumption that items will have fairly low demands, with means near 0.

Recall that we previously had assumed that $C > \max_i \sigma_i \delta_i^U$. Because of the new parameterization, we will revise this assumption to be $C^- > \max_i \sigma_i \delta_i^L$ and $C^+ > \max_i \sigma_i \delta_i^U$.

Returning to the problem at hand, because of the non-identical cost parameters, in the first extreme point $\hat{\boldsymbol{\varepsilon}}$, nature would choose ε to solve the following LP relaxation of a knapsack problem:

$$\begin{aligned} Z_1 = \max_{\boldsymbol{\varepsilon}} \sum_{i=1}^{n} h_i \sigma_i \varepsilon_i \\ \text{subject to } 0 \leq \varepsilon_i \leq \delta_i^L, \\ \sum_{i=1}^{n} \sigma_i \varepsilon_i \leq C^-. \end{aligned} \tag{4.8}$$

Note, however, that in an extreme point of type (2), item i experiences its maximum demand. Hence, we remove item i from the LP and obtain the following LP

$$\begin{aligned} Z_2(i) = \max_{\boldsymbol{\varepsilon}} \sum_{j \neq i} h_j \sigma_j \varepsilon_j \\ \text{subject to } 0 \leq \varepsilon_j \leq \delta_j^L, \\ \sum_{j \neq i} \sigma_j \varepsilon_j \leq C^-. \end{aligned} \tag{4.9}$$

We now define a parameter η_i which represents the difference in the two objective functions normalized by the product of the holding cost of item i and the standard deviation of item i. That is, $\eta_i = \frac{Z_1 - Z_2(i)}{h_i \sigma_i}$, and using this notation, let

$$s_i^* = \mu_i + \frac{b_i \delta_i^U - h_i \eta_i}{b_i + h_i} \sigma_i.$$

Before proceeding, recall that we previously defined the following two functions:

$$f(\boldsymbol{s}) = \max_{\boldsymbol{\varepsilon}} \sum_{i=1}^{n} \max \left\{ h_i \left(s_i - \mu_i - \varepsilon_i \sigma_i \right), b_i(\mu_i + \varepsilon_i \sigma_i - s_i) \right\} \tag{4.10}$$

$$g(\boldsymbol{s}, \boldsymbol{\varepsilon}) = \sum_{i=1}^{n} \max \left\{ h_i \left(s_i - \mu_i - \varepsilon_i \sigma_i \right), b_i(\mu_i + \varepsilon_i \sigma_i - s_i) \right\}. \tag{4.11}$$

Proposition 4.6 *Suppose that for the n item case under non-identical demand and non-identical cost parameters, that $\frac{\min_i b_i}{\max_i h_i} > n-1$. Then $s_i^* = \mu_i + \frac{b_i\delta_i^U - h_i\eta_i}{b_i+h_i}\sigma_i$, $\forall\ i$ is optimal.*

Proof. Observe that if the value of the coefficient $h_i\sigma_i$ is small, then the stock level of item i will be relatively large as nature would prefer to utilize its budget for small demands on other items. However, if the value of $h_i\sigma_i$ is large, then the stock level of item i will be relatively small as the importance of protecting against holding costs is increased. Note also that this particular set of stock levels, $\boldsymbol{s}^*$, was chosen to make nature indifferent between the two previously described extreme points, as we now show. We now extend the definition of $g(\boldsymbol{s}, \boldsymbol{\varepsilon})$ from the previous discussion to be

$$g(\boldsymbol{s}, \boldsymbol{\varepsilon}) = \sum_{i=1}^{n} \max\left\{h_i(s_i - \mu_i - \varepsilon_i\sigma_i), b_i(\mu_i + \varepsilon_i\sigma_i - s_i)\right\}, \tag{4.12}$$

for a given set of actions, $\boldsymbol{\varepsilon}$, for nature and a given set of stock levels chosen by the seller, $\boldsymbol{s}$. Then for the first extreme point,

$$\begin{aligned} g(\boldsymbol{s}^*, \hat{\boldsymbol{\varepsilon}}) = \sum_{i=1}^{n} h_i(s_i^* - \mu_i) + \max_{\boldsymbol{\varepsilon}^-} \sum_{i=1}^{n} h_i\sigma_i\varepsilon_i^- \\ \text{subject to} \sum_{i=1}^{n} \varepsilon_i^-\sigma_i \le C^-, \\ 0 \le \varepsilon_i^- \le \delta_i^L\ \forall\ i, \\ = \sum_{i=1}^{n} h_i(s_i^* - \mu_i) + Z_1, \end{aligned}$$

and for the other extreme point,

$$\begin{aligned} g(\boldsymbol{s}^*, \tilde{\boldsymbol{\varepsilon}}(i)) = b_i\left(\mu_i + \sigma_i\delta_i^U - s_i^*\right) + \sum_{j\neq i} h_j(s_j^* - \mu_j) + \max_{\boldsymbol{\varepsilon}^-} \sum_{j\neq i} h_j\sigma_j\varepsilon_j^- \\ \text{subject to} \sum_{j\neq i} \varepsilon_j^-\sigma_j \le C^-, \\ 0 \le \varepsilon_j^- \le \delta_j^L\ \forall\ j \neq i. \end{aligned}$$

Recall that $Z_2(i) = \max_{\boldsymbol{\varepsilon}^-} \sum_{j \neq i} h_j \sigma_j \varepsilon_j^-$. We substitute $Z_2(i)$ into the objective function to obtain

$$\begin{aligned}
g(\boldsymbol{s^*}, \tilde{\boldsymbol{\varepsilon}}(i)) &= b_i \left(\mu_i + \sigma_i \delta_i^U - s_i^*\right) + \sum_{j \neq i} h_j (s_j^* - \mu_j) + Z_2(i) \\
&= b_i \left(\sigma_i \delta_i^U - \frac{b_i \delta_i^U - h_i \eta_i}{b_i + h_i} \sigma_i\right) + \sum_{j \neq i} h_j (s_j^* - \mu_j) + Z_2(i) \\
&= b_i \left(\frac{h_i \sigma_i \delta_i^U + h_i \eta_i}{b_i + h_i} \sigma_i\right) + \sum_{j \neq i} h_j (s_j^* - \mu_j) + Z_2(i) \\
&= h_i \left(\frac{b_i \sigma_i \delta_i^U + b_i \eta_i}{b_i + h_i} \sigma_i\right) + \sum_{j \neq i} h_j (s_j^* - \mu_j) + Z_2(i) \\
&= h_i \left(\frac{b_i \sigma_i \delta_i^U - h_i \eta_i + (b_i + h_i)\eta_i}{b_i + h_i} \sigma_i\right) + \sum_{j \neq i} h_j (s_j^* - \mu_j) + Z_2(i) \\
&= h_i \left(\frac{b_i \sigma_i \delta_i^U - h_i \eta_i}{b_i + h_i} \sigma_i\right) + \sum_{j \neq i} h_j (s_j^* - \mu_j) + Z_2(i) + h_i \eta_i \sigma_i \\
&= h_i \left(\mu_i + \frac{b_i \sigma_i \delta_i^U - h_i \eta_i}{b_i + h_i} \sigma_i - \mu_i\right) + \sum_{j \neq i} h_j (s_j^* - \mu_j) + Z_1 \\
&= h_i \left(s_i^* - \mu_i\right) + \sum_{j \neq i} h_j (s_j^* - \mu_j) + Z_1 \\
&= g(\boldsymbol{s^*}, \hat{\boldsymbol{\varepsilon}}).
\end{aligned}$$

Because our choice of i is arbitrary, the cost under action $\hat{\boldsymbol{\varepsilon}}$ is identical to the cost under action $\tilde{\boldsymbol{\varepsilon}}(i)$ for any choice of i. We must also show that no other extreme point for nature can inflict a higher cost. However, note that the analysis above showed that each item's contribution to cost was chosen so that it was identical under $\hat{\varepsilon}_i$ and $\varepsilon_i = \delta_i^U$. Therefore, any number of items can experience their maximal demand without affecting the cost to the seller. As a result, the cost to the seller is equivalent to the objective value of the following mathematical program where the constraint on C^+ has been relaxed:

$$
\begin{aligned}
\max_{\boldsymbol{\varepsilon}} \sum_{i=1}^{n} & \max\left\{h_i\left(s_i^* - (\mu_i + \sigma_i\varepsilon_i)\right), b_i\left((\mu_i + \sigma_i\varepsilon_i) - s_i^*\right)\right\} \\
\text{subject to} & \sum_{i=1}^{n} \varepsilon_i^- \sigma_i \le C^-, \\
& 0 \le \varepsilon_i^- \le \delta_i^L \ \forall\, i, \\
& 0 \le \varepsilon_i^+ \le \delta_i^U \ \forall\, i, \\
& \varepsilon_i = \varepsilon_i^+ - \varepsilon_i^- \ \forall\, i,
\end{aligned}
$$

which is an upper bound on the optimal solution for nature for the given set of stock levels $\boldsymbol{s}^*$. Since this solution attains its upper bound, it must be optimal.

Suppose that the seller would like to set $\overline{\boldsymbol{s}}$ as the stock levels with $\overline{s}_i \ge s_i^* \ \forall\, i$. But since extreme point (1) associates smaller demands with every item, nature could select extreme point (1) to penalize every item's increase in stock levels. This would result in an increase of $\sum_{i=1}^{n} h_i\left(\overline{s}_i - s_i^*\right) > 0$ in the cost to the seller.

Now consider $\boldsymbol{s} \ne \boldsymbol{s}^*$ where there exists at least some index i such that $s_i < s_i^*$. Observe that there must also exist an index i' such that $s_{i'}^* - s_{i'}$ is maximal. That is, i' is such that $s_{i'}^* - s_{i'} \ge s_j^* - s_j \ \forall\, j \ne i$. Observe that nature could simply choose the extreme point $\tilde{\varepsilon}(i')$ to inflict cost $g(\boldsymbol{s}^*, \tilde{\boldsymbol{\varepsilon}}(i')) + b_{i'}(s_{i'}^* - s_{i'}) - \sum_{j \ne i'} h_j(s_j^* - s_j)$. Recall that we have assumed that $\min_i b_i > (n-1)\max_i h_i$ and i' is the index for which $s_{i'}^* - s_{i'}$ is maximal. Then it must be that

$$
\begin{aligned}
b_{i'}(s_{i'}^* - s_{i'}) - \sum_{j \ne i'} h_j(s_j^* - s_j) &\ge b_{i'}(s_{i'}^* - s_{i'}) - \sum_{j \ne i'} h_j(s_{i'}^* - s_{i'}) \\
&\ge b_{i'}(s_{i'}^* - s_{i'}) - \sum_{j \ne i'} \max_i h_i(s_{i'}^* - s_{i'}) \\
&\ge \left(\min_i b_i - (n-1)\max_i h_i\right)(s_{i'}^* - s_{i'}) \\
&> 0.
\end{aligned}
$$

Therefore, $\boldsymbol{s}^*$ is optimal. Note that other than assuming $C^+ > \max_i \sigma_i \delta_i^U$, this analysis was independent of the values of C^-, C^+. However, the analysis is completely dependent upon the assumption that $\min_i b_i > (n-1) \max_i h_i$. In the analysis above, without the condition that $\min_i b_i > (n-1) \max_i h_i$, there may exist a direction v that allows the seller to lower the aggregate holding cost by enough to offset the increase in potential backorder costs. As mentioned previously, while this may be a reasonable assumption for a small number of items, as n grows larger, this assumption may be too strong.

Similarly, the idea of breaking the non-negative real line into disjoint intervals does not provide insight into the general case. However, for relatively small fixed values of n, this idea can be applied relatively efficiently. Unfortunately, the number of intervals is a function of the orderings of $\sigma_i \delta_i^L, \sigma_i \delta_i^U$, and their partial sums. As a result, the number of intervals grows combinatorially as a function of n, preventing this analysis from being applied for large n. Instead, we proceed under the assumption that the number of items n is sufficiently large so that $\frac{\min_i b_i}{\max_i h_i} < n - 1$ and obtain some useful bounds through Lagrangian relaxation.

Additionally, we make further assumptions on the budgets of uncertainty available to nature. Recall that under very small or large budgets in the previous section, the problem for the seller is trivial. We also feel that this does not accurately represent an operational environment as these budgets would correspond to extremely high and extremely low variance demands. We therefore proceed assuming that the budgets are moderate in size, since that will still permit nature to have a certain degree of latitude to inflict additional costs to the seller. We state this assumption explicitly in the following section.

4.4 Lagrangian Relaxation

Observe that the joint constraints related to the budgets of uncertainty are what make the problem difficult. Put another way, if these constraints were relaxed, the problem would separate by item and we could analyze each item separately. Therefore, we apply Lagrangian relaxation to this problem, assuming that at least one of the joint constraints will be active in an optimal solution.

Turning to our original multi-item problem, recall that the optimal action for the seller is found by solving the n item problem below:

$$Z = \min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} \sum_{i=1}^{n} \max\left\{h_i(s_i - \mu_i - \varepsilon_i\sigma_i), b_i(\mu_i + \varepsilon_i\sigma_i - s_i)\right\}$$

subject to

$$\sum_{i=1}^{n} \varepsilon_i^{+}\sigma_i \leq C^{+} = \mathbb{E}\left[\tilde{Z}^{+}\right] + \delta_Z^{+}\sqrt{\mathrm{Var}\left(\tilde{Z}^{+}\right)}, \tag{1}$$

$$\sum_{i=1}^{n} \varepsilon_i^{+}\sigma_i \leq C^{-} = \left|\mathbb{E}\left[\tilde{Z}^{-}\right]\right| + \delta_Z^{-}\sqrt{\mathrm{Var}\left(\tilde{Z}^{+}\right)}, \tag{2}$$

$$0 \leq \varepsilon_i^{+} \leq \delta_i^{U} \;\forall\; i. \tag{3}$$

$$0 \leq \varepsilon_i^{-} \leq \delta_i^{L} \;\forall\; i \tag{4}$$

$$\varepsilon_i = \varepsilon_i^{+} - \varepsilon_i^{-} \;\forall\; i. \tag{5}$$

Associate Lagrange multipliers of λ_{+} and λ_{-} with constraints (1) and (2) respectively. Then the following is a Lagrangian relaxation of the original problem:

$$q\left(\lambda_{+}, \lambda_{-}\right) = \min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} \quad \begin{array}{l} \sum_{i=1}^{n} \left[\max\left\{ h_i(s_i - (\mu_i + \varepsilon_i\sigma_i)), b_i((\mu_i + \varepsilon_i\sigma_i) - s_i) \right\} \right] \\ +\lambda_{+}\left(C^{+} - \sum_{i=1}^{n}\sigma_i\varepsilon_i^{+}\right) + \lambda_{-}\left(C^{-} - \sum_{i=1}^{n}\sigma_i\varepsilon_i^{-}\right) \end{array}$$

subject to

$$0 \leq \varepsilon_i^{+} \leq \delta_i^{U} \;\forall\; i,$$

$$0 \leq \varepsilon_i^{-} \leq \delta_i^{L} \;\forall\; i,$$

$$\varepsilon_i = \varepsilon_i^{+} - \varepsilon_i^{-} \;\forall\; i.$$

It can easily be seen that $q(\lambda_{+}, \lambda_{-})$ forms an upper bound on the cost to the seller. Any feasible action by nature in the original problem remains feasible in the Lagrangian problem, but the cost to the seller is increased by the slack in each of the joint budget constraints associated with their Lagrange multipliers. Hence, $q(\lambda_{+}, \lambda_{-}) \geq Z$.

Distributing the Lagrange multipliers, incorporating the resulting terms into the maximization, and substituting $\varepsilon_i = \varepsilon_i^+ - \varepsilon_i^-$, we see that

$$q(\lambda_+, \lambda_-) = \min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} \quad \sum_{i=1}^{n} \max \quad \left\{ \begin{array}{c} h_i(s_i - \mu_i) + (h_i - \lambda_-)\sigma_i \varepsilon_i^- - (h_i + \lambda_+)\sigma_i \varepsilon_i^+, \\ b_i(\mu_i - s_i) - (b_i + \lambda_-)\sigma \varepsilon_i^- + (b_i - \lambda_+)\varepsilon_i^+ \sigma_i \end{array} \right\}$$
$$+(C^+\lambda_+ + C^-\lambda_-).$$

Given the vector of stock levels $\boldsymbol{s}$ and multipliers λ_+, λ_-, observe that each ε_i^+ and ε_i^- should be at its maximum when its coefficient is positive. Otherwise, for negative coefficients, ε_i^+ and ε_i^- should each be set to 0. Consider the term associated with the holding cost inside the maximization. Observe that it is optimal for nature to set $\varepsilon_i^+ = 0 \ \forall \ i$ and $\varepsilon_i^- = \delta_i^L$ if $h_i > \lambda_-$ for this term. Next, consider the term associated with the backorder cost. When considering the backorder cost, we consider the sign of $b_i - \lambda_+$. That is, if $\lambda_+ < b_i$, then we should have $\varepsilon_i^+ = \delta_i^U$ and $\varepsilon_i^+ = 0$ otherwise. Additionally, when considering the backorder costs, note that it is optimal for nature to set $\varepsilon_i^- = 0 \ \forall \ i$ since $b_i + \lambda_+ > 0$.

Based on these observations and through the use of indicator variables, we rewrite the objective function

$$q(\lambda_+, \lambda_-) = \min_{\boldsymbol{s}} \sum_{i=1}^{n} \max \quad \left\{ \begin{array}{c} h_i(s_i - \mu_i) + \sigma_i \delta_i^L (h_i - \lambda_-) \mathbb{1}_{\{h_i > \lambda_-\}}, \\ b_i(\mu_i - s_i) + \sigma_i \delta_i^U (b_i - \lambda_+) \mathbb{1}_{\{b_i > \lambda_+\}} \end{array} \right\} + (C^+\lambda_+ + C^-\lambda_-).$$

Since the joint constraints have been relaxed, the problem separates by item into independent single-item subproblems. Therefore, we proceed by considering each item individually. Then, as before in the single item case, we minimize each element of the sum by setting the two expressions of the maximization equal to each other. This yields

$$s_i^* = \mu_i + \left[\frac{\delta_i^U (b_i - \lambda_+)}{b_i + h_i} \mathbb{1}_{\{b_i > \lambda_+\}} - \frac{\delta_i^L (h_i - \lambda_-)}{b_i + h_i} \mathbb{1}_{\{h_i > \lambda_-\}} \right] \sigma_i. \tag{4.13}$$

Note that this solution depends on the value of the Lagrange multipliers not only through the indicators, but also in the numerators of the fractions. Note also that these stock levels must be bounded. That is,

$$\mu_i - \frac{h_i \delta_i^L}{b_i + h_i}\sigma_i \le s_i^* \le \mu_i + \frac{b_i \delta_i^U}{b_i + h_i}\sigma_i,$$

with the lower bound attained for $\lambda_+ > b_i, \lambda_- = 0$ and the upper bound attained for $\lambda_+ = 0, \lambda_- > h_i$.

Returning to the cost function, we substitute this inventory level into the Lagrangian relaxation objective function to obtain

$$q(\lambda_+,\ \lambda_-) = \\ \sum_{i=1}^{n}\left[\frac{h_i}{b_i + h_i}\sigma_i\delta_i^U(b_i - \lambda_+)\mathbb{1}_{\{b_i > \lambda_+\}} + \frac{b_i}{b_i + h_i}\sigma_i\delta_i^L(h_i - \lambda_-)\mathbb{1}_{\{h_i > \lambda_-\}}\right] + (C^+\lambda_+ + C^-\lambda_-).$$

Recall that because constraints on nature were relaxed, the above expression is an upper bound on the total cost to the seller for any values of $\lambda_+, \lambda_- \ge 0$. Then in order to obtain the tightest possible upper bound, we minimize $q(\lambda_+, \lambda_-)$ with respect to λ_+, λ_-. First, observe that the Lagrangian objective function can be rewritten as the sum of two independent functions with each associated with one of the Lagrange multipliers, λ_+ and λ_- :

$$q(\lambda_+, \lambda_-) = \begin{array}{l} \displaystyle\sum_{i=1}^{n}\left[\frac{h_i}{b_i + h_i}\sigma_i\delta_i^U(b_i - \lambda_+)\mathbb{1}_{\{b_i > \lambda_+\}}\right] + C^+\lambda_+ \\ \displaystyle + \sum_{i=1}^{n}\left[\frac{b_i}{b_i + h_i}\sigma_i\delta_i^L(h_i - \lambda_-)\mathbb{1}_{\{h_i > \lambda_-\}}\right] + C^-\lambda_- \end{array}.$$

Then the tightest possible bound is given by

$$q\left(\lambda_+^*, \lambda_-^*\right) = \begin{array}{l} \displaystyle\min_{\lambda_+ \ge 0}\left\{\sum_{i=1}^{n}\left[\frac{h_i}{b_i + h_i}\sigma_i\delta_i^U(b_i - \lambda_+)\mathbb{1}_{\{b_i > \lambda_+\}}\right] + C^+\lambda_+\right\} \\ \displaystyle + \min_{\lambda_- \ge 0}\left\{\sum_{i=1}^{n}\left[\frac{b_i}{b_i + h_i}\sigma_i\delta_i^L(h_i - \lambda_-)\mathbb{1}_{\{h_i > \lambda_-\}}\right] + C^-\lambda_-\right\} \end{array}.$$

We take the derivative of $q(\lambda_+, \lambda_-)$ with respect to λ_+ and obtain

$$\frac{\partial q}{\partial \lambda_+} = C^+ - \sum_{i=1}^{n}\left[\frac{h_i}{b_i + h_i}\sigma_i\delta_i^U\mathbb{1}_{\{b_i > \lambda_+\}}\right], \tag{4.14}$$

for $\lambda_+ \notin \{b_i\}_{i=1}^n$. For $\lambda_+ \in \{b_i\}_{i=1}^n$, q is not differentiable. Hence, for these points, we define the derivative of q to be the right-hand derivative: $\frac{\partial_+}{\partial\lambda_+}q$, where

$$\frac{\partial_+}{\partial\lambda_+}q(\lambda_+,\lambda_-) := \lim_{h\to 0^+}\frac{q(\lambda_+ + h,\lambda_-) - q(\lambda_+,\lambda_-)}{h}.$$

Similarly, the derivative of $q(\lambda_+,\lambda_-)$ with respect to λ_- is

$$\frac{\partial q}{\partial\lambda_-} = C^- - \sum_{i=1}^n\left[\frac{b_i}{b_i+h_i}\sigma_i\delta_i^L\mathbb{1}_{\{h_i>\lambda_-\}}\right]. \tag{4.15}$$

for $\lambda_- \notin \{h_i\}_{i=1}^n$. Again, for these points $\lambda_- \in \{h_i\}_{i=1}^n$, we will define the derivative of q to be the right-hand derivative: $\frac{\partial_+ q}{\partial\lambda_-}$.

Note that both derivatives are piecewise constant functions that are non-decreasing in λ_+ and λ_-, respectively. Thus we can find the optimal choices of λ_+ and λ_- easily by searching over the set of possible optimal values, which we describe now. First, note that $\frac{\partial q}{\partial\lambda_+}$ is a constant for all values of λ_+. Then the optimal choice of λ_+, λ_+^*, satisfies:

$$\begin{aligned}&\text{(i)} \quad \tfrac{\partial q}{\partial\lambda_+} > 0, \text{ if } \lambda_+ \geq \lambda_+^*, \quad \text{and}\\ &\text{(ii)} \quad \tfrac{\partial q}{\partial\lambda_+} \leq 0, \text{ if } \lambda_+ < \lambda_+^*.\end{aligned} \tag{4.16}$$

where as before, the derivative at non-differentiable points is defined the be the right-hand derivative.

Naturally, λ_-^* must satisfy a similar condition.

Note that the set of optimal multipliers is not necessarily unique. Recall that from (4.14) and (4.15), the derivative is piecewise constant. Therefore, if there exists a range of λ_+, λ_- values for which $\frac{\partial q}{\partial\lambda_+} = 0$ or $\frac{\partial q}{\partial\lambda_-} = 0$, then there exist multiple optimal values for λ_+, λ_-. This additionally results in the existence of multiple sets of optimal stock levels for the seller. However, note that the original function q is convex in λ_+ as well as λ_-. Therefore, while there may exist multiple optimal values of λ_+, λ_-, all such solutions result in the same value of $q(\lambda_+,\lambda_-)$. That is, while there may be more than one optimal set of Lagrange multipliers (which would result in multiple optimal vectors of stock levels), there is a unique value of

q associated with all of the optimal Lagrange multipliers. As a result, we will choose these multipliers as follows:

$$\begin{aligned} &\text{(i)} \quad \lambda_+^* = \sup\left\{\lambda_+ \geq 0 | \tfrac{\partial q}{\partial \lambda_+} \leq 0\right\}, \quad \text{and} \\ &\text{(ii)} \quad \lambda_-^* = \inf\left\{\lambda_- \geq 0 | \tfrac{\partial q}{\partial \lambda_-} \geq 0\right\}. \end{aligned} \tag{4.17}$$

However, because of the way we have defined our conditions above, we are able to select a unique set of Lagrange multipliers. Moreover, because we are selecting the largest possible optimal value of λ_+ and the smallest possible optimal value of λ_-, these conditions correspond to the optimal action by the seller with minimal stock levels for all items as can be seen in (4.13).

Recall that the derivatives are piecewise constant, and thus the value of the derivatives will only change at the non-differentiable points. Then, the only points that need to be considered are the elements of the set $\{0, \{b_i\}_{i=1}^n\}$. Note also that if $C^+ > \sum_{i=1}^n \frac{h_i}{b_i+h_i}\sigma_i\delta_i^U$, the derivative with respect to λ_+ is strictly positive and $\lambda_+^* = 0$ minimizes $q(\lambda_+, \lambda_-)$.

Observe that a similar argument with respect to λ_- holds, and the set of possible values to consider for λ_-^* are the elements of the set $\{0, \{h_i\}_{i=1}^n\}$. Additionally, if $C^- > \sum_{i=1}^n \frac{b_i}{b_i+h_i}\sigma_i\delta_i^L$, then the derivative with respect to λ_- is strictly positive and $\lambda_-^* = 0$ minimizes $q(\lambda_+, \lambda_-)$.

This ensures that we can always efficiently find a unique set of values for λ_+ and λ_- under any given set of parameters and budget for nature, which result in a unique set of stock levels, $\boldsymbol{s}^*$, through (4.13) and (4.17). For the remainder of this discussion, we will refer to these stock levels as the Lagrangian policy, which as shown above is feasible and easily computed.

Recall that we assumed nature's budgets of uncertainty are moderate in size. Specifically, we will assume any half of the items can experience either their minimal or maximal demands.

That is, we will assume:

$$\text{(i)} \quad \max_{Q:|Q|=\lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^U \leq C^+ < \sum_{i=1}^n \sigma_i \delta_i^U \text{ and}$$
$$\text{(ii)} \quad \max_{Q:|Q|=\lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^L \leq C^- < \sum_{i=1}^n \sigma_i \delta_i^L$$

for the remainder of our discussion of the general multi-item problem. Now note that the previous analysis was with respect to the fully generalized problem. Our new assumptions on the budget available to nature and the relationship among the item cost parameters give rise to additional structure in nature's response and the stock levels selected by the Lagrangian policy. We first note that by assumption, $C^- \geq \max_{Q:|Q|=\lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^L$. Then it follows that the value of λ_-^* is bounded from above. That is, for large value of λ_-, the majority of the indicator functions found in (4.15) would evaluate to 0, which would violate the conditions found in (4.17). We make this more explicit in the following Lemma.

Lemma 4.1 *Without loss of generality, assume that the set of holding costs* $\{h_i\}_{i=1}^n$ *is in increasing order with* $h_i \leq h_{i+1}$*. Under the assumption that* $b_i > h_i \ \forall \ i$ *and that* $C^- \geq \max_{Q:|Q|=\lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^L$*, the optimal choice of* λ_- *must lie in the set* $\{0\} \bigcup \{h_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor}$*.*

Proof. We first show that for $\lambda_- \geq h_{\lfloor \frac{n}{2} \rfloor}$, the derivative in (4.15) is bounded below as follows

$$\begin{aligned}
\frac{\partial q}{\partial \lambda -} &= C^- - \sum_{i=1}^n \frac{b_i}{b_i + h_i} \sigma_i \delta_i \mathbb{1}_{\{h_i > \lambda_-\}} \\
&= C^- - \sum_{i=\lfloor \frac{n}{2} \rfloor}^n \frac{b_i}{b_i + h_i} \sigma_i \delta_i^L \\
&> C^- - \sum_{i=\lfloor \frac{n}{2} \rfloor}^n \sigma_i \delta_i^L \\
&\geq C^- - \max_{Q:|Q|=\lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^L \\
&\geq 0.
\end{aligned}$$

Then it must be that the derivative is positive for all $\lambda_- \geq h_{\lfloor \frac{n}{2} \rfloor}$. Thus, no value of λ_- larger than $h_{\lfloor \frac{n}{2} \rfloor}$ can be optimal. ∎

Note also that we can utilize our assumption on the budget for large demands, C^+, to impose additional structure on the optimal Lagrangian multiplier λ_+^*.

Lemma 4.2 *Under the assumptions that $b_i > h_i \ \forall\ i$ and $C^+ \geq \max_{Q:|Q|=\lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^U$, the policy suggested by Lagrangian relaxation is such that $\sum_{i=1}^n \sigma_i \varepsilon_i^+ < C^+$.*

Proof. Note that

$$\frac{\partial q}{\partial \lambda_+} = C^+ - \sum_{i=1}^n \left[\frac{h_i}{b_i + h_i} \sigma_i \delta_i^U \mathbb{1}_{\{b_i > \lambda_+\}} \right],$$

and recall that we have assumed that $C^+ \geq \max_{Q:|Q|=\lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^U$. Since we can simply choose the $\lceil \frac{n}{2} \rceil$ items with largest $\sigma_i \delta_i^U$, it must also be true that $C^+ \geq \sum_{i=1}^n \frac{1}{2} \sigma_i \delta_i^U$, as each item in the chosen set Q dominates another item excluded from the set. Now observe that because $b_i > h_i \ \forall\ i$, $\frac{h_i}{b_i + h_i} < \frac{1}{2}$. Hence,

$$\begin{aligned} \sum_{i=1}^n \left[\frac{h_i}{b_i + h_i} \sigma_i \delta_i^U \mathbb{1}_{\{b_i > \lambda_+\}} \right] &\leq \sum_{i=1}^n \left[\frac{h_i}{b_i + h_i} \sigma_i \delta_i^U \right] \\ &< \sum_{i=1}^n \left[\frac{1}{2} \sigma_i \delta_i^U \right] \leq C^+. \end{aligned}$$

As a result, it must be that the $\frac{\partial}{\partial \lambda_+} q \geq 0 \ \forall\ \lambda_+ > 0$, and, hence, the derivative must be strictly positive. Therefore, under our assumptions, $\lambda_+^* = 0$.

Then by complementary slackness, the associated constraint must not be tight, and it follows that under the Lagrangian policy, nature's corresponding optimal action, $\hat{\boldsymbol{\varepsilon}}$, must be such that

$$C^+ > \sum_{i=1}^n \sigma_i \hat{\varepsilon}_i^+.$$

∎

Recall from the previous discussion that the Lagrangian relaxation yields an upper bound on the true cost since a constraint associated with the inner maximization was relaxed. However, we cannot yet establish how well this policy performs relative to the optimal policy. In order to evaluate the performance of this policy, we must also obtain a lower bound on the true cost. Before doing so, we first examine the choice of stock levels to the seller, and establish upper and lower bounds on the optimal stock levels.

Recall that the analysis in the previous sections utilized the trade-off between potential holding and backorder costs in order to set stock levels for each item. We again appeal to these ideas to prove the bounds on the optimal stock levels. We begin by introducing some notation and making two observations:

1. Denote $\overline{s}_i = \mu_i + \dfrac{b_i \delta_i^U}{b_i + h_i}\sigma_i$. We demonstrate below that at a stock level of $\overline{s}_i$, the cost to the seller associated with item i is identical for actions $\varepsilon_i = \delta_i^U$ or $\varepsilon_i = 0$. That is,

$b_i(\mu_i - \overline{s}_i + \delta_i^U \sigma_i) = h_i(\overline{s}_i - \mu_i)$.

2. Denote $\underline{s}_i = \mu_i + \dfrac{b_i \delta_i^U - h_i \delta_i^L}{b_i + h_i}\sigma_i$. We demonstrate below that at a stock level of $\underline{s}_i$, the cost to the seller associated with item i is identical for actions $\varepsilon_i = \delta_i^U$ or $\varepsilon_i = -\delta_i^L$. That is,

$b_i(\mu_i - \underline{s}_i + \delta_i^U \sigma_i) = h_i(\underline{s}_i - \mu_i + \delta_i^L \sigma_i)$.

(4.18)

We demonstrate in the following lemmas that these values are carefully chosen. As we show, these values provide upper and lower bounds on the optimal stock levels for the seller. We begin by considering the upper bound. Before proceeding, we introduce the following two sets:

$$\begin{aligned} \overline{\mathcal{S}}_+ &= \left\{ i \;\middle|\; s_i > \overline{s}_i, \varepsilon_i > 0 \right\}, \text{ and} \\ \overline{\mathcal{S}}_- &= \left\{ i \;\middle|\; s_i > \overline{s}_i, \varepsilon_i \le 0 \right\}. \end{aligned} \tag{4.19}$$

First, observe that if the seller chooses a stock level of $\overline{s}_i = \mu_i + \frac{b_i}{b_i+h_i}\sigma_i\delta_i$ for item i, the costs arising from item i to the seller are identical under nature's actions of $\varepsilon_i = 0$ or $\varepsilon_i = \delta_i^U$, both $\frac{b_i h_i}{b_i + h_i}\sigma_i \delta_i^U$, respectively:

$$\begin{aligned} h_i\left(\overline{s}_i - (\mu_i + 0)\right) &= h_i\left(\frac{b_i}{b_i + h_i}\sigma_i\delta_i^U\right) \\ &= b_i\left(\frac{h_i}{b_i + h_i}\sigma_i\delta_i^U + \overline{s}_i - \overline{s}_i\right) \\ &= b_i\left(\mu_i + \frac{b_i + h_i}{b_i + h_i}\sigma_i\delta_i^U - \overline{s}_i\right) \\ &= b_i\left(\mu_i + \sigma_i\delta_i^U - \overline{s}_i\right). \end{aligned}$$

Note that an intermediate value $0 < \varepsilon_i < \delta_i^U$ cannot maximize the cost to the seller. It is easy to see that for a given value of ε_i and stock level s_i, either $h_i(s_i - \mu_i - \varepsilon_i\sigma_i)$ or $b_i(\mu_i + \varepsilon_i\sigma_i - s_i)$ is maximal. Then if the holding cost is maximal, the cost can be increased by choosing $\varepsilon_i = 0$ and if the backorder cost is maximal, the cost can be increased by choosing $\varepsilon_i = \delta_i^U$.

From this observation, we can infer that nature would prefer to set $\varepsilon_i \leq 0$ for all items $i \in \overline{\mathcal{S}} = \overline{\mathcal{S}}_+ \bigsqcup \overline{\mathcal{S}}_-$, where $\bigsqcup$ is the disjoint union, since doing so increases the cost to the seller. We will make this more precise in the following lemma.

Lemma 4.3 *Consider a set of stock levels $\boldsymbol{s}$ and define sets $\overline{\mathcal{S}}_+$ and $\overline{\mathcal{S}}_-$ as in (4.19) with respect to $\boldsymbol{s}$. Then under any optimal set of actions by nature, $\left|\overline{\mathcal{S}}_+\right| = 0$.*

Proof. Suppose that the claim were not true and there existed an optimal action for nature, $\boldsymbol{\varepsilon}(\boldsymbol{s})$, where $\left|\overline{\mathcal{S}}_+\right| > 0$. Consider the effect on the seller if nature were to change its action for item $i, i \in \overline{\mathcal{S}}_+$, to $\varepsilon_i = 0$. Note that the cost contribution of item i to the seller would now be:

$$
\begin{aligned}
h_i(s_i - \mu_i) &> h_i(\overline{s}_i - \mu_i) \\
&= b_i(\mu_i + \delta_i^U \sigma_i - \overline{s}_i) \\
&\geq b_i(\mu_i + \varepsilon_i(\boldsymbol{s})\sigma_i - \overline{s}_i) \\
&> b_i(\mu_i + \varepsilon_i(\boldsymbol{s})\sigma_i - s_i),
\end{aligned}
$$

since $i \in \underline{\mathcal{S}}$.

Therefore, the cost to the seller would strictly increase if nature were to change its action for item $i \in \overline{\mathcal{S}}_+$ from $\varepsilon(\boldsymbol{s}) > 0$ to $\varepsilon_i = 0$. This is always feasible since it does not consume any of the budgets of uncertainty. Therefore, the original action $\boldsymbol{\varepsilon}(\boldsymbol{s})$ must be suboptimal and $\left|\overline{\mathcal{S}}_+\right| = 0$. ∎

Having characterized nature's actions through Lemma 4.3, we can characterize the seller's actions through the following lemma.

Lemma 4.4 *The optimal inventory level s_i^* is bounded above by $\overline{s}_i = \mu_i + \frac{b_i}{b_i+h_i}\sigma_i\delta_i^U$.*

Proof. Suppose that the claim were not true and there existed an instance for which $s_i^* > \overline{s}_i$ for at least one item i. Consider the effect of lowering the stock level of each item $i \in \overline{\mathcal{S}}$.

From Lemma 4.3, we know that for each item $i \in \overline{\mathcal{S}}, \varepsilon_i(\boldsymbol{s^*}) \leq 0$. Because each item $i \in \overline{\mathcal{S}}$ has a stock level where $s_i^* > \overline{s}_i$, there exists some $\eta > 0$ such that $s_i^* - \eta > \overline{s}_i \ \forall \ i \in \overline{\mathcal{S}}$. Then define the following set of stock levels, $\boldsymbol{s}'$ in the following manner:

$$
s_i' = \begin{cases} s_i^*, & \text{if } i \notin \overline{\mathcal{S}} \\ s_i^* - \eta, & \text{if } i \in \overline{\mathcal{S}} \end{cases}.
$$

Note that by construction $s_i' > \overline{s}_i \iff i \in \overline{\mathcal{S}}$ and consider nature's optimal action for stock levels $\boldsymbol{s}'$, $\boldsymbol{\varepsilon}(\boldsymbol{s}')$. Again, utilizing Lemma 4.3, for every item $i \in \overline{\mathcal{S}}$, $\varepsilon_i \leq 0$. Observe that

$$
f(\boldsymbol{s}') = g(\boldsymbol{s}', \boldsymbol{\varepsilon}(\boldsymbol{s}')),
$$

and consider

$$g(\boldsymbol{s}^*, \boldsymbol{\varepsilon}(\boldsymbol{s}^{'})) - g(\boldsymbol{s}^{'}, \boldsymbol{\varepsilon}(\boldsymbol{s}^{'})) = \sum_{i \in \overline{\mathcal{S}}} h_i (s_i^* - s_i^{'})$$

$$> 0.$$

Then it must be that, $f(\boldsymbol{s}^{'}) = g(\boldsymbol{s}^{'}, \boldsymbol{\varepsilon}(\boldsymbol{s}^{'})) < g(\boldsymbol{s}^*, \boldsymbol{\varepsilon}(\boldsymbol{s}^{'})) \leq f(\boldsymbol{s}^*)$, contradicting our assumption that the set of stock levels s^* were optimal. ∎

Note that the proofs of Lemma 4.3 and Lemma 4.4 did not utilize our assumption that $\min_i b_i > \max_i h_i$, and hence hold in general.

Next, we turn to our lower bound. As we have just done, we will first characterize the response of nature before proving the lower bound. As stated previously in (4.18), at an inventory level of

$$\underline{s}_i = \mu_i + \frac{b_i \delta_i^U - h_i \delta_i^L}{b_i + h_i} \sigma_i,$$

the cost to the seller associated with item i is identical for actions $\varepsilon_i = \delta_i^U$ and $\varepsilon_i = -\delta_i^L$, and as before, if these actions are feasible, intermediate values are suboptimal. That is,

$$\begin{aligned}
b_i(\mu_i + \delta_i^U \sigma_i - \underline{s}_i) &= b_i \left(\frac{(b_i + h_i)\delta_i^U}{b_i + h_i} \sigma_i - \frac{b_i \delta_i^U - h_i \delta_i^L}{b_i + h_i} \sigma_i \right) \\
&= b_i \left(\frac{h_i \delta_i^U + h_i \delta_i^L}{b_i + h_i} \sigma_i \right) \\
&= h_i \left(\frac{b_i \delta_i^U + b_i \delta_i^L}{b_i + h_i} \sigma_i \right) \\
&= h_i \left(\frac{b_i \delta_i^U + b_i \delta_i^L + h_i \delta_i^L - h_i \delta_i^L}{b_i + h_i} \sigma_i \right) \\
&= h_i \left(\frac{b_i \delta_i^U - h_i \delta_i^L}{b_i + h_i} \sigma_i + \delta_i^L \sigma_i \right) \\
&= h_i \left(\frac{b_i \delta_i^U - h_i \delta_i^L}{b_i + h_i} \sigma_i + \mu_i - \mu_i + \delta_i^L \sigma_i \right) \\
&= h_i \left(\underline{s}_i - \mu_i + \delta_i^L \sigma_i \right).
\end{aligned}$$

Then it must be that for any item i with stock level $s_i < \underline{s}_i$, action $\varepsilon_i = \delta_i^U$ maximizes the cost to the seller.

Utilizing this observation, we characterize nature's actions through the following lemma. Before proceeding, we first introduce some more notation:

$$\begin{aligned}
\underline{\mathcal{S}}_{+} &= \left\{ i \;\middle|\; s_i \le \underline{s}_i, \varepsilon_i > 0 \right\}, \\
\underline{\mathcal{S}}_{-} &= \left\{ i \;\middle|\; s_i \le \underline{s}_i, \varepsilon_i \le 0 \right\}, \\
\underline{\mathcal{S}}_{+}^{C} &= \left\{ i \;\middle|\; s_i > \underline{s}_i, \varepsilon_i > 0 \right\}, \text{ and} \\
\underline{\mathcal{S}}_{-}^{C} &= \left\{ i \;\middle|\; s_i > \underline{s}_i, \varepsilon_i \le 0 \right\}.
\end{aligned} \tag{4.20}$$

We will also refer to sets $\underline{\mathcal{S}} = \underline{\mathcal{S}}_{+} \bigsqcup \underline{\mathcal{S}}_{-}$ and $\underline{\mathcal{S}}^{C} = \{1, \dots, n\} \setminus \underline{\mathcal{S}}$.

Lemma 4.5 *Define $\underline{\mathcal{S}}_{+}, \underline{\mathcal{S}}_{-}, \underline{\mathcal{S}}_{+}^{C}, \underline{\mathcal{S}}_{-}^{C}$ as in* (4.20) *with respect to a given stock level, $\hat{\mathbf{s}}$, and an optimal action for nature, $\boldsymbol{\varepsilon}^{*})(\hat{\mathbf{s}})$. Assuming*

1. $C^{+} \ge \max_{|Q| = \lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^U$ *and*
2. $C^{-} \ge \max_{|Q| = \lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^L$*,*

if $\varepsilon^{*}(\hat{\mathbf{s}})$ is optimal for the set of stock levels $\hat{\mathbf{s}}$, then $|\underline{\mathcal{S}}_{+}^{C}| \le \left(\lceil \frac{n}{2} \rceil - |\underline{\mathcal{S}}|\right)^{+}$. That is, if $|\underline{\mathcal{S}}| \ge \lceil \frac{n}{2} \rceil$, then $\left|\underline{\mathcal{S}}_{+}^{C}\right| = 0$.

Recall that for all items $i \in \underline{\mathcal{S}}^{C}$, an action of $\varepsilon_i = -\delta_i^L$ (if feasible) maximizes the cost to the seller. This lemma demonstrates that nature will not set $\varepsilon_i > 0$ for any item $i \in \underline{\mathcal{S}}^{C}$ unless $|\underline{\mathcal{S}}| < \lceil \frac{n}{2} \rceil$, in which case, every item in $\underline{\mathcal{S}}$ will experience its maximum demand. Put another way, if $|\underline{\mathcal{S}}| \le \lceil \frac{n}{2} \rceil$, then there exists an optimal action by nature where $\varepsilon_i = \delta_i^U \; \forall \; i \in \underline{\mathcal{S}}$, and as a result, $\underline{\mathcal{S}}_{-}$ is empty.

Proof. Suppose that the lemma were not true and that there existed an optimal action for nature, $\boldsymbol{\varepsilon}^{*}$, where $|\underline{\mathcal{S}}_{+}^{C}| > \left(\lceil \frac{n}{2} \rceil - |\underline{\mathcal{S}}|\right)^{+}$. Next, we examine the size of $|\underline{\mathcal{S}}_{-}|$ through the following two cases:

Case 1. $|\underline{\mathcal{S}}_-| = 0$ and hence, $|\underline{\mathcal{S}}_+| = |\underline{\mathcal{S}}|$. That is, $\varepsilon_i^* > 0 \ \forall \ i \in \underline{\mathcal{S}}$.

Case 2. $|\underline{\mathcal{S}}_-| > 0$. That is, there exists at least one item $i \in \underline{\mathcal{S}}$ with $\varepsilon_i^* \leq 0$.

Proof under Case 1

Note that in this case, our assumption becomes $|\underline{\mathcal{S}}_+| + |\underline{\mathcal{S}}_+^C| > \lceil \frac{n}{2} \rceil$ and hence $|\underline{\mathcal{S}}_-| + |\underline{\mathcal{S}}_-^C| < \lfloor \frac{n}{2} \rfloor$. But recall that for any item $i \in \underline{\mathcal{S}}^C$, the item's cost contribution to the seller is maximized when $\varepsilon_i = -\delta_i^L$. Next note that by assumption, $\underline{\mathcal{S}}_+^C$ is nonempty, and from the argument above, $|\underline{\mathcal{S}}_-| + |\underline{\mathcal{S}}_-^C| < \lfloor \frac{n}{2} \rfloor$. Then combining this observation with our assumption on C^-, for any one item $j \in \underline{\mathcal{S}}_+^C$, we have that

$$\sum_{i \in \underline{\mathcal{S}}_-} |-\delta_i^L \sigma_i| + \sum_{i \in \underline{\mathcal{S}}_-^C} |-\delta_i^L \sigma_i| + |-\delta_j^L \sigma_j| \leq C^-.$$

As a result, there is sufficient budget for nature to change ε_j for any item j in $\underline{\mathcal{S}}_+^C$ from the previous value of $\varepsilon_j^*(\hat{\boldsymbol{s}})$ to $-\delta_j^L$, which would strictly increase the cost to the seller. Therefore, the original action by nature must have been suboptimal, yielding a contradiction.

Proof under Case 2

As we have noted previously, the action that maximizes the cost contribution of item i to the seller for $i \in \underline{\mathcal{S}}$ is $\varepsilon_i = \delta_i^U$. Conversely, the action that maximizes the cost contribution of item i to the seller for $i \in \underline{\mathcal{S}}^C$ is $\varepsilon_i = -\delta_i^L$. Then we pose the following question: If $\boldsymbol{\varepsilon^*}$ is optimal, why is it that $\underline{\mathcal{S}}_-$ and $\underline{\mathcal{S}}_+^C$ are nonempty? Utilizing the same idea as in Case 1, we make the following claims:

(i) $|\underline{\mathcal{S}}_-| + |\underline{\mathcal{S}}_-^C| \geq \lceil \frac{n}{2} \rceil$, and

(ii) $|\underline{\mathcal{S}}_+| + |\underline{\mathcal{S}}_+^C| \geq \lceil \frac{n}{2} \rceil$.

Beginning with claim (i), recall that for any item $i \in \underline{\mathcal{S}}^C$, its contribution to the seller's cost is maximized at $\varepsilon_i = -\delta_i^L$. That is, $h_i(\hat{s}_i - \mu_i + \delta_i^L \sigma_i) > b_i(\mu_i + \delta_i^U \sigma_i - \hat{s}_i) \ \forall \ i \in \underline{\mathcal{S}}^C$.

Therefore, if $\underline{\mathcal{S}}_+^C$ is nonempty, it must be the case that nature does not have sufficient budget to set $\varepsilon_i = -\delta_i^L$ for any item $i \in \underline{\mathcal{S}}_+^C$. However, recall that we assumed that C^- was relatively large. That is, $C^- \geq \max_{|Q|=\lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^L$. Therefore, it must be that $|\underline{\mathcal{S}}_-| + |\underline{\mathcal{S}}_-^C| \geq \lceil \frac{n}{2} \rceil$, proving claim (i).

Claim (ii) is proven in a similar way. Recall that for any item $i \in \underline{\mathcal{S}}$, its contribution to the seller's cost is maximized at $\varepsilon_i = \delta_i^U$. That is,

$$h_i(\hat{s}_i - \mu_i + \delta_i^L \sigma_i) \leq b_i(\mu_i + \delta_i^U \sigma_i - \hat{s}_i) \ \forall \ i \in \underline{\mathcal{S}},$$

with equality if and only if $\hat{s}_i = \underline{s}_i$.

Now suppose that $\hat{s}_i = \underline{s}_i$ for some item $i \in \underline{\mathcal{S}}_-$. Recall from claim (i) that if $\underline{\mathcal{S}}_+^C$ is nonempty, then $|\underline{\mathcal{S}}_+| + |\underline{\mathcal{S}}_+^C| \leq \lfloor \frac{n}{2} \rfloor$. As a result, it is feasible for nature to set $\varepsilon_i = \delta_i^U$ for at least one item $i \in \underline{\mathcal{S}}_-$ where $\hat{s}_i = \underline{s}_i$, transferring these items from $\underline{\mathcal{S}}_-$ to $\underline{\mathcal{S}}_+$. Note that this can be done without decreasing the total cost to the seller:

$$\begin{aligned} b_i(\mu_i + \delta_i^U \sigma_i - \underline{s}_i) &= h_i(\underline{s}_i - \mu_i + \delta_i^L \sigma_i) \\ &\geq h_i(\underline{s}_i - \mu_i + |\varepsilon_i^*(\hat{\boldsymbol{s}})\sigma_i|) \ \forall \ i \in \underline{\mathcal{S}}_-, \hat{s}_i = \underline{s}_i. \end{aligned}$$

Note also that this transfer action does not affect the set $\underline{\mathcal{S}}_+^C$, which will remain nonempty. As a result, this action can be repeated as necessary. Therefore, we proceed assuming (without loss of generality) that every item i where $\hat{s}_i = \underline{s}_i$ is an element of $\underline{\mathcal{S}}_+$. Then, if $\underline{\mathcal{S}}_-$ is nonempty, and $\hat{s}_i < \underline{s}_i \ \forall \ i \in \underline{\mathcal{S}}_-$, it must be because nature does not have sufficient budget to set $\varepsilon_i = \delta_i^U$ for every item $i \in \underline{\mathcal{S}}$. However, recall that we assumed that $C^+ \geq \max_{|Q|=\lceil \frac{n}{2} \rceil} \sum_{i \in Q} \sigma_i \delta_i^U$. Therefore, it must be that $|\underline{\mathcal{S}}_+| + |\underline{\mathcal{S}}_+^C| \geq \lceil \frac{n}{2} \rceil$, proving claim (ii).

Having proven claims (i) and (ii), now note that if n is odd,

$$|\underline{\mathcal{S}}_-| + |\underline{\mathcal{S}}_-^C| + |\underline{\mathcal{S}}_+| + |\underline{\mathcal{S}}_+^C| \geq 2 \left\lceil \frac{n}{2} \right\rceil = n + 1,$$

yielding an immediate contradiction.

Now suppose instead that n is even. Then in order for both claims (i) and (ii) to hold, we must have

$$|\underline{\mathcal{S}}_{-}| + |\underline{\mathcal{S}}_{-}^{C}| = \left\lceil \frac{n}{2} \right\rceil = \frac{n}{2}$$
$$\text{and } |\underline{\mathcal{S}}_{+}| + |\underline{\mathcal{S}}_{+}^{C}| = \left\lceil \frac{n}{2} \right\rceil = \frac{n}{2}.$$

Recall that in this case $\underline{\mathcal{S}}_{-}$ and $\underline{\mathcal{S}}_{+}^{C}$ are nonempty. Then we can select some item $i_1 \in \underline{\mathcal{S}}_{-}$ and some item $i_2 \in \underline{\mathcal{S}}_{+}^{C}$ and set $\varepsilon_{i_1} = \delta_{i_1}^{U}$, $\varepsilon_{i_2} = -\delta_{i_2}^{L}$ without altering any values of ε_j^* where $j \neq i_1, i_2$ as there will still be exactly $\frac{n}{2}$ items with $\varepsilon_i \leq 0$ and $\frac{n}{2}$ items with $\varepsilon_i > 0$. Then because of our assumption on the budgets of uncertainty, this proposed action by nature must be feasible. Denote this new action by nature to be $\bar{\boldsymbol{\varepsilon}}$. Then because

$$b_{i_1}(\mu_{i_1} + \delta_{i_1}^{U}\sigma_{i_1} - \hat{s}_{i_1}) > h_{i_1}(\hat{s}_{i_1} - \mu_{i_1} + \delta_{i_1}^{L}\sigma_{i_1}) \geq h_{i_1}(\hat{s}_{i_1} - \mu_{i_1} + \varepsilon_{i_1}^{*}\sigma_{i_1}),$$
$$\text{and } h_{i_2}(\hat{s}_{i_2} - \mu_{i_2} + \varepsilon_{i_2}\sigma_{i_2}) > b_{i_2}(\mu_{i_2} + \delta_{i_2}^{U}\sigma_{i_2} - \hat{s}_{i_2}) \ \forall \ \varepsilon_{i_2} > 0,$$

it must be that

$$g\left(\hat{\boldsymbol{s}}, \bar{\boldsymbol{\varepsilon}}\right) - g\left(\hat{\boldsymbol{s}}, \boldsymbol{\varepsilon}^{*}\right) > 0,$$

and changing nature's response from $\boldsymbol{\varepsilon}^*$ to $\bar{\boldsymbol{\varepsilon}}$ has strictly increased the cost to the seller, contradicting the assumption that $\boldsymbol{\varepsilon}^*$ is an optimal set of actions for nature. ∎

Having characterized nature's actions, we are now ready to prove a lower bound on the optimal stock level.

Proposition 4.7 *Under the assumptions*

1. $C^{+} \geq \max_{|S|=\lceil n/2 \rceil} \sum_{i \in S} \sigma_i \delta_i^{U}$,
2. $C^{-} \geq \max_{|S|=\lceil n/2 \rceil} \sum_{i \in S} \sigma_i \delta_i^{L}$, *and*
3. $\min_i b_i > \max_i h_i$,

the optimal stock level for each item i, s_i^*, is bounded below by $\underline{s}_i = \mu_i + \frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i$.

Proof. Recall that the value $\underline{s}_i = \mu_i + \frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i$ is the point at which the item's cost contribution to the seller for actions $\varepsilon_i = \delta_i^U$ and $\varepsilon_i = -\delta_i^L$ are equivalent, both yielding a cost of $\frac{b_ih_i(\delta_i^U+\delta_i^L)}{b_i+h_i}\sigma_i$. Recall also that as C^+, C^- become sufficiently large and the joint constraints become non-binding, $\underline{s}_i = \mu_i + \frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i$ is optimal. In this section, we will prove through contradiction the validity of the lower bound. We will suppose the existence of an optimal solution violating our bound, then construct another solution and an action for nature which contradict the original optimality assumption.

Suppose that some set of stock levels $\boldsymbol{s}^*$ is optimal with at least one item below our proposed lower bound. That is, there is at least one item i such that $s_i^* < \underline{s}_i = \mu_i + \frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i$. Next, define the sets $\underline{\mathcal{S}}_+, \underline{\mathcal{S}}_-$, etc. as before in (4.20) with respect to $\boldsymbol{s}^*$. Now consider the following set of stock levels $\hat{\boldsymbol{s}}$ where we have raised the stock levels of the items in $\underline{\mathcal{S}}$ up to our proposed lower bound. Hence, for all items i, we now have that

$$\hat{s}_i = s_i^* \vee \underline{s}_i.$$

Next, let $\boldsymbol{\varepsilon}(\hat{\boldsymbol{s}})$ be an optimal action for nature when the seller chooses stock levels $\hat{\boldsymbol{s}}$. Then because $\boldsymbol{s}^*$ is assumed to be optimal, and $\boldsymbol{\varepsilon}(\hat{\boldsymbol{s}})$ may not be an optimal action by nature for the set of stock levels $\boldsymbol{s}^*$, it must be that

$$g(\hat{\boldsymbol{s}}, \boldsymbol{\varepsilon}^*(\hat{\boldsymbol{s}})) = f(\hat{\boldsymbol{s}}) \geq f(\boldsymbol{s}^*) \geq g(\boldsymbol{s}^*, \boldsymbol{\varepsilon}^*(\hat{\boldsymbol{s}})).$$

Below, we will demonstrate that $f(\hat{\boldsymbol{s}}) < f(\boldsymbol{s}^*)$, contradicting our assumption that $\boldsymbol{s}^*$ is optimal.

However, let us characterize the form of $\boldsymbol{\varepsilon}^*(\hat{\boldsymbol{s}})$. Note that there is not necessarily a single optimal action for nature. As we have seen in the analysis of the two-item case, we can easily have several actions by nature which result in the same total cost to the seller.

Now examining the items $i \in \underline{\mathcal{S}}$, recall that for the set of stock levels $\hat{\boldsymbol{s}}$, the inventory levels for items $i \in \underline{\mathcal{S}}$ have been raised to $\underline{s}_i$, which is the point where both actions $\varepsilon_i = \delta_i^U$ and

$\varepsilon_i = -\delta_i^L$ maximize the cost to the seller. Furthermore, for each item i in the complementary set, $\underline{\mathcal{S}}^C$, $\hat{s}_i > \underline{s}_i$, and hence, for these items, action $\varepsilon_i = -\delta_i^L$ maximizes the cost to the seller. Using these observations, we will use Lemma 4.5 to construct an optimal action for nature.

We proceed by examining two different cases:

Case 1. $|\underline{\mathcal{S}}| \geq \lceil \frac{n}{2} \rceil$

Case 2. $|\underline{\mathcal{S}}| < \lceil \frac{n}{2} \rceil$

Proof of Case 1

Denote the set $\underline{\mathcal{S}}_{max}$ to be a set of $\lceil \frac{n}{2} \rceil$ items $i \in \underline{\mathcal{S}}$ with maximal $\hat{s}_i - s_i^*$. Put another way, if we suppose without loss of generality that the items $i \in \underline{\mathcal{S}}$ are ordered by $\hat{s}_i - s_i^*$ in decreasing order, then $\underline{\mathcal{S}}_{max}$ would be the first $\lceil \frac{n}{2} \rceil$ items. That is, by construction $\underline{\mathcal{S}}_{max}$ must satisfy

$$\hat{s}_i - s_i^* \geq \hat{s}_j - s_j^* \ \forall\ i \in \underline{\mathcal{S}}_{max}, j \in \underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max}.$$

Next, note that by construction, $|\underline{\mathcal{S}}_{max}| = \lceil \frac{n}{2} \rceil$. Note also that $\underline{\mathcal{S}}^C \bigsqcup (\underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max})$ contains all items not in $\underline{\mathcal{S}}_{max}$. Therefore, $\left|\underline{\mathcal{S}}^C\right| + |\underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max}| = \lfloor \frac{n}{2} \rfloor$. Then because of our assumptions on C^+, C^-, it is feasible for nature to set:

$$\begin{aligned}
\hat{\varepsilon}_i &= -\delta_i^L \ \forall\ i \in \underline{\mathcal{S}}^C \\
\hat{\varepsilon}_i &= \quad \delta_i^U \ \forall\ i \in \underline{\mathcal{S}}_{max} \\
\hat{\varepsilon}_i &= -\delta_i^L \ \forall\ i \in \{\underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max}\}\,.
\end{aligned} \tag{4.21}$$

We argue that our proposed action by nature maximizes the cost contribution of each item to the seller with respect to the set of stock levels $\hat{\boldsymbol{s}}$, forming an optimal response by nature. Observe that each item $i \in \underline{\mathcal{S}}^C$ experiences minimal demand, maximizing the cost to the seller. Next, recall that items $i \in \underline{\mathcal{S}}$ have had their stock levels raised to $\hat{s}_i = \underline{s}_i$ at which either action δ_i^U or $-\delta_i^L$ is optimal for nature. Now note that every item $i \in \underline{\mathcal{S}}$ experiences either a maximal or minimal demand. Thus, our proposed actions also maximize the cost contribution for all items in the set of items $\underline{\mathcal{S}}$. Since the cost contribution of each item to the seller is maximized, our proposed actions must be an optimal set of actions for nature when the seller chooses the set of stock levels $\hat{\boldsymbol{s}}$. We now denote this proposed optimal action as $\boldsymbol{\varepsilon}(\hat{\boldsymbol{s}})$. We next calculate the cost of our supposedly optimal solution $\boldsymbol{s}^*$ when nature employs the set of actions $\boldsymbol{\varepsilon}(\hat{\boldsymbol{s}})$. This cost is

$$g(\boldsymbol{s}^*, \boldsymbol{\varepsilon}(\hat{\boldsymbol{s}})) = f(\hat{\boldsymbol{s}}) + \sum_{i \in \underline{\mathcal{S}}_{max}} b_i(\hat{s}_i - s_i^*) - \sum_{j \in \{\underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max}\}} h_j(\hat{s}_j - s_j^*) \tag{4.22}$$

$$= g(\hat{\boldsymbol{s}}, \boldsymbol{\varepsilon}(\hat{\boldsymbol{s}})) + \sum_{i \in \underline{\mathcal{S}}_{max}} b_i(\hat{s}_i - s_i^*) - \sum_{j \in \{\underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max}\}} h_j(\hat{s}_j - s_j^*). \tag{4.23}$$

But recall that it was assumed $\min_i b_i > \max_i h_i$ and that by construction, $\hat{s}_i - s_i \geq \hat{s}_j - s_j^*, \ \forall \ i \in \underline{\mathcal{S}}_{max}, j \in \underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max}$. Therefore, each element of the first sum dominates each element of the second sum. As a result, we can lower bound (4.22) by substituting $\alpha = \max_{j \in \{\underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max}\}} h_j(\hat{s}_j - s_j)$ for each term in both sums. Note further that because of construction, $|\underline{\mathcal{S}}_{max}| = \lceil \frac{n}{2} \rceil$ and as a result, $|\underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max}| \leq \lfloor \frac{n}{2} \rfloor$. Then it must be that

$$\begin{aligned} g(\boldsymbol{s}^*, \hat{\boldsymbol{\varepsilon}}) >& f(\hat{\boldsymbol{s}}) + \sum_{i \in \underline{\mathcal{S}}_{max}} \alpha - \sum_{i \in \{\underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max}\}} \alpha \\ =& f(\hat{\boldsymbol{s}}) + \left\lceil \frac{n}{2} \right\rceil \alpha - |\underline{\mathcal{S}} \setminus \underline{\mathcal{S}}_{max}| \alpha \\ >& \ f(\hat{\boldsymbol{s}}), \end{aligned}$$

and we have obtained our contradiction.

Proof of Case 2

Having proven the claim for the case where $|\underline{\mathcal{S}}| \geq \lceil \frac{n}{2} \rceil$, we must now prove the same claim for the case where $|\underline{\mathcal{S}}| < \lceil \frac{n}{2} \rceil$.

Recall that in (4.21), $\hat{\varepsilon}_i = -\delta_i^L \ \forall \ i \in \underline{\mathcal{S}}^C$. In this case, $|\underline{\mathcal{S}}^C|$ may be large enough that this is no longer feasible. However, recall that this is not critical to our argument. For our argument to hold, we simply need to demonstrate that there is an optimal action for nature in which at least half of the items $i \in \underline{\mathcal{S}}$ experience incremental backorder costs.

Now note that the result of Lemma 4.5 implies that in the event that $|\underline{\mathcal{S}}| < \lceil \frac{n}{2} \rceil$, $\varepsilon_i = \delta_i^U \; \forall \; i \in \underline{\mathcal{S}}$. Because of the size of the set $\underline{\mathcal{S}}$, Lemma 4.5 implies that $|\underline{\mathcal{S}}| + |\underline{\mathcal{S}}_+^C| \leq \lceil \frac{n}{2} \rceil$. Consequently, because of our assumption on the size of the budget C^+, it is feasible for nature to set $\varepsilon_i = \delta_i^U$ for all items $i \in \underline{\mathcal{S}}$ and any other actions taken in this set are suboptimal. As a result,

$$g(\boldsymbol{s}^*, \hat{\boldsymbol{\varepsilon}}) = f(\hat{\boldsymbol{s}}) + \sum_{i \in \underline{\mathcal{S}}} b_i(\hat{s}_i - s_i^*) > f(\hat{\boldsymbol{s}}),$$

which completes the proof. ■

Now that we have produced lower and upper bounds on the optimal stock levels, we are ready to demonstrate a lower bound on the optimal cost to the seller through the following proposition.

Proposition 4.8 *The optimal cost to the seller, $f(\boldsymbol{s}^*)$, is bounded below by*

$$f(\boldsymbol{s}^*) \geq \sum_{i=1}^{n} h_i \frac{b_i\delta_i^U - h_i\delta_i^L}{b_i + h_i}\sigma_i + \frac{1}{2}h_i\sigma_i\delta_i^L.$$

Proof. Note that as a consequence of the lower and upper bounds on the set of optimal stock levels, the original problem to the seller:

$$\begin{aligned} Z = \min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} & \sum_{i=1}^{n} \max\{h_i(s_i - \mu_i - \varepsilon_i\sigma_i), b_i(\mu_i + \varepsilon_i\sigma_i - s_i)\} \\ & \text{subject to } \boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \boldsymbol{\delta_Z}) \end{aligned} \tag{4.24}$$

is equivalent to the following mathematical program:

$$\begin{aligned} \overline{Z} = \min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} & \sum_{i=1}^{n} \max\{h_i(s_i - \mu_i - \varepsilon_i\sigma_i), b_i(\mu_i + \varepsilon_i\sigma_i - s_i)\} \\ & \text{subject to } \boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \boldsymbol{\delta_Z}) \\ & \quad\quad \underline{s}_i \leq s_i \leq \overline{s}_i \; \forall \; i. \end{aligned} \tag{4.25}$$

Next, we utilize the max-min inequality which states that for any two arbitrary, non-empty sets X, Y and any function $h : X \times Y \to \mathbb{R}$,

$$\sup_{x \in X} \inf_{y \in Y} h(x, y) \leq \inf_{y \in Y} \sup_{x \in X} h(x, y)$$

(see Lemma 36.1 of reference [19]). Note, however, that in (4.25), the set $U(\boldsymbol{\delta}, \boldsymbol{\delta_Z})$ and the set $\{\boldsymbol{s} : \underline{s}_i \leq s_i \leq \overline{s}_i \ \forall\ i\}$ are both compact sets. Hence, in our problem setting, the inequality can be strengthened to

$$\max_{x \in X} \min_{y \in Y} h(x, y) \leq \min_{y \in Y} \max_{x \in X} h(x, y).$$

As a consequence of this inequality, we are able to lower bound $\overline{Z}$ with the value of the following mathematical program:

$$\begin{aligned} W = \max_{\boldsymbol{\varepsilon}} \min_{\boldsymbol{s}} & \sum_{i=1}^{n} \max \{h_i(s_i - \mu_i + \varepsilon_i \sigma_i), b_i(\mu_i + \varepsilon_i \sigma_i - s_i)\} \\ \text{subject to } & \boldsymbol{\varepsilon} \in U(\boldsymbol{\delta}, \boldsymbol{\delta_Z}) \\ & \underline{s}_i \leq s_i \leq \overline{s}_i \ \forall\ i, \end{aligned} \tag{4.26}$$

with the relationship $Z = \overline{Z} \geq W$. Now, we bound W by assuming that nature adopts the following policy: $\bar{\boldsymbol{\varepsilon}}$ where $\bar{\varepsilon}_i = -\frac{1}{2}\delta_i^L \ \forall\ i$. In almost all cases, this policy is suboptimal for nature. That is, the policy is optimal for nature only when $s_i = \overline{s}_i \ \forall\ i, \delta_i^L = \delta_j^L, h_i = h_j \ \forall\ i,\ \forall\ j$. Although this policy is likely to be suboptimal, it provides us with an easily analyzed expression. In any case, given our assumption on the budget for negative deviations, C^-, such a policy is always feasible. Then under this assumed policy for nature, the problem facing the seller with respect to the problem in (4.26) is:

$$\begin{aligned} \underline{W} = \min_{\boldsymbol{s}} & \sum_{i=1}^{n} \max \left\{ h_i \left(s_i - \mu_i + \frac{1}{2}\delta_i^L \sigma_i \right), b_i \left(\mu_i - \frac{1}{2}\delta_i^L \sigma_i - s_i \right) \right\} \\ \text{subject to} \quad & \underline{s}_i \leq s_i \leq \overline{s}_i \ \forall\ i. \end{aligned} \tag{4.27}$$

Next, observe that because of the bounds on the stock levels, the problem can be further simplified. Consequently, for any set of stock levels, $\boldsymbol{s}$, satisfying $\underline{s}_i \leq s_i \leq \overline{s}_i \ \forall\ i$, the quantity $\mu_i - \frac{1}{2}\delta_i^L\sigma_i - s_i < 0 \; \forall \; i$ and hence $s_i - \mu_i + \frac{1}{2}\delta_i^L\sigma_i > 0 \; \forall \; i$. As a result, each item will incur a holding cost. Thus, the problem simplifies to:

$$\begin{aligned} \underline{W} = \min_{\boldsymbol{s}} & \sum_{i=1}^{n} h_i \left(s_i - \mu_i + \frac{1}{2}\delta_i^L\sigma_i \right) \\ \text{subject to} \qquad & \underline{s}_i \leq s_i \leq \overline{s}_i \; \forall \; i, \end{aligned} \tag{4.28}$$

which has the trivial solution $s_i^* = \underline{s}_i \; \forall \; i$ with cost $\underline{W} = \sum_{i=1}^{n} h_i \frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i + \frac{1}{2}h_i\sigma_i\delta_i^L$. Note that this implies the following relation:

$$f(\boldsymbol{s}^*) = Z = \overline{Z} \geq W \geq \underline{W} = \sum_{i=1}^{n} h_i \frac{b_i\delta_i^U - h_i\delta_i^L}{b_i + h_i}\sigma_i + \frac{1}{2}h_i\sigma_i\delta_i^L. \tag{4.29}$$

Thus $f(\boldsymbol{s}^*) \geq \sum_{i=1}^{n} h_i \frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i + \frac{1}{2}h_i\sigma_i\delta_i^L$. ∎

Now that we have found a lower bound on the true optimal cost, $f(\boldsymbol{s}^*)$, we can proceed with our analysis of the performance of the Lagrangian relaxation policy.

Theorem 4.1 *The policy suggested by the Lagrangian relaxation is a 2-approximation of the optimal policy under assumptions*

1. $C^+ \geq \max_{|S|=\lceil n/2 \rceil} \sum_{i \in S} \sigma_i\delta_i^U$,

2. $C^- \geq \max_{|S|=\lceil n/2 \rceil} \sum_{i \in S} \sigma_i\delta_i^L$,

3. $\min_i b_i > \max_i h_i$, *and*

4. $\delta_i^U \geq \delta_i^L \; \forall \; i$.

Proof. Recall that the cost under the Lagrangian policy is upper bounded by the Lagrangian relaxation for the choice of $(0, 0)$ as our Lagrange multipliers (λ_+, λ_-). That is,

$$
\begin{aligned}
q\left(\lambda_+^*, \lambda_-^*\right) &\le q(0,0) \\
&= \sum_{i=1}^{n} \left[\frac{h_i}{b_i + h_i} \sigma_i \delta_i^U (b_i - 0) \mathbb{1}_{\{b_i > 0\}} \right] + \sum_{i=1}^{n} \left[\frac{b_i}{b_i + h_i} \sigma_i \delta_i^L (h_i - 0) \mathbb{1}_{\{h_i > 0\}} \right] + 0 \\
&= \sum_{i=1}^{n} h_i \left[\frac{b_i}{b_i + h_i} \sigma_i \left(\delta_i^U + \delta_i^L \right) \right]
\end{aligned}
$$

since $(0,0)$ is not necessarily an optimal set of multipliers. Denote the optimal set of stock levels for Problem (4.24) as $\boldsymbol{s}^*$ and recall that by Proposition 4.8, the optimal cost to the seller is bounded below by

$$
f(\boldsymbol{s}^*) \ge \underline{W} = \sum_{i=1}^{n} \left[h_i \frac{b_i \delta_i^U - h_i \delta_i^L}{b_i + h_i} \sigma_i \right] + \frac{1}{2} \sum_{i=1}^{n} \left[h_i \sigma_i \delta_i^L \right].
$$

Factoring out the h_i term, we see that

$$
\underline{W} = \sum_{i=1}^{n} h_i \left[\frac{b_i \delta_i^U - h_i \delta_i^L}{b_i + h_i} \sigma_i + \frac{1}{2} \delta_i^L \sigma_i \right].
$$

Next, observe that for these two quantities:

$$
\begin{aligned}
\frac{\partial}{\partial b_i} q(0,0) &= \frac{h_i^2 \sigma_i (\delta_i^U + \delta_i^L)}{(b_i + h_i)^2} > 0 \ \forall\ i, \ \text{and} \\
\frac{\partial}{\partial b_i} \underline{W} &= \frac{h_i^2 \sigma_i (\delta_i^U + \delta_i^L)}{(b_i + h_i)^2} > 0 \ \forall\ i.
\end{aligned}
$$

We define the ratios $A = q(\lambda_+^*, \lambda_-^*)/Z$ and $\overline{A} = q(0,0)/\underline{W}$. Note that by definition, $A \le \overline{A}$. Next, differentiating $\overline{A}$ with respect to b_i, we see that

$$
\begin{aligned}
\frac{\partial}{\partial b_i} \overline{A} &= \frac{\underline{W} \frac{\partial}{\partial b_i} q(0,0) - q(0,0) \frac{\partial}{\partial b_i} \underline{W}}{\underline{W}^2} \\
&= \frac{\frac{h_i^2 \sigma_i (\delta_i^U + \delta_i^L)}{(b_i + h_i)^2}}{\underline{W}^2} \left(\underline{W} - q(0,0) \right) \\
&< 0,
\end{aligned}
$$

since $q(0,0) \ge Z \ge \underline{W}$. Hence, we can upper bound the ratio $\overline{A}$ by decreasing the backorder cost, b_i, for any item i.

We can make the same argument with respect to the quantity δ_i^U. That is,

$$\frac{\partial}{\partial \delta_i^U} q(0,0) = \frac{b_i h_i}{b_i + h_i}\sigma_i > 0 \ \forall \ i$$
$$\frac{\partial}{\partial \delta_i^U} \underline{W} = \frac{b_i h_i}{b_i + h_i}\sigma_i > 0 \ \forall \ i,$$

and, as a result,

$$\begin{aligned}
\frac{\partial}{\partial \delta_i^U} \overline{A} &= \frac{\underline{W}\frac{\partial}{\partial \delta_i^U} q(0,0) - q(0,0)\frac{\partial}{\partial \delta_i^U}\underline{W}}{\underline{W}^2} \\
&= \frac{\frac{b_i h_i}{b_i + h_i}}{\underline{W}^2}\left(\underline{W} - q(0,0)\right) \\
&< 0.
\end{aligned}$$

Hence, we can upper bound the ratio $\overline{A}$ by decreasing the quantity δ_i^U for any item i.

Recall from our earlier assumptions that $\min_i b_i > \max_i h_i$ and therefore $b_i > h_i \ \forall \ i$. Recall also that it was assumed $\delta_i^U \geq \delta_i^L \ \forall \ i$. Hence, there exist two constants $\alpha \geq 1, \beta \geq 1$ such that $b_i \geq \alpha h_i \ \forall \ i, \delta_i^U \geq \beta \delta_i^L \ \forall \ i$. Since $\frac{\partial}{\partial b_i}\overline{A} < 0$ and $\frac{\partial}{\partial \delta_i^U}\overline{A}$, by decreasing b_i, δ_i^U, for any item i, we are able to upper bound $\overline{A}$. We decrease each b_i to its lower bound αh_i and decrease each δ_i^U to its lower bound $\beta \delta_i^L$, and obtain

$$\begin{aligned}
\overline{A} &\leq \frac{\sum_{i=1}^{n} h_i \left[\frac{\alpha h_i}{\alpha h_i + h_i}\sigma_i\left(\beta\delta_i^L + \delta_i^L\right)\right]}{\sum_{i=1}^{n} h_i \left[\frac{\alpha\beta h_i \delta_i^L - h_i\delta_i^L}{\alpha h_i + h_i}\sigma_i + \frac{1}{2}\delta_i^L\sigma_i\right]} \\
&= \frac{\frac{\alpha(1+\beta)}{\alpha+1}\sum_{i=1}^{n} h_i\sigma_i}{\sum_{i=1}^{n} h_i\left[\frac{2\alpha\beta h_i\delta_i^L - 2h_i\delta_i^L + (h_i\alpha + h_i)\delta_i^L}{2(\alpha h_i + h_i)}\sigma_i\right]} \\
&= \frac{\frac{\alpha(1+\beta)}{\alpha+1}\sum_{i=1}^{n} h_i\sigma_i}{\left[\frac{2\alpha\beta - 1 + \alpha}{2(\alpha+1)}\right]\sum_{i=1}^{n} h_i\sigma_i\delta_i^L} \\
&= \frac{2\alpha\beta + 2\alpha}{2\alpha\beta + \alpha - 1}.
\end{aligned}$$

This quantity attains a maximum of 2 when $\alpha = 1, \beta = 1$. Hence,

$$\frac{q(\lambda_+^*, \lambda_-^*)}{Z} \leq \frac{2\alpha\beta + 2\alpha}{2\alpha\beta + \alpha - 1} \leq 2,$$

and the cost of the policy resulting from the Lagrangian relaxation at most a factor of 2 larger than the optimal cost. ∎

The above analysis also allows us to gain a better understanding of the policy's dependence upon the parameters α and β. Note that as $\alpha \to \infty$, the ratio approaches $\frac{2\beta+2}{2\beta+1} \leq \frac{4}{3}$ and hence, our performance guarantee improves to $\frac{4}{3}$. Additionally, note that as $\beta \to \infty$, the ratio approaches 1 and the Lagrangian policy becomes optimal.

Note, however, that the above analysis used the quantity $\underline{W}$, which provides a loose lower bound on the true optimal cost since we assumed a suboptimal policy for nature to arrive at a lower bound to Problem (4.26). Note that this bound of 2 appears in the analysis when $b_i = h_i, \delta_i^U = \delta_i^L \ \forall\, i$. This can be traced to the assumption made in the calculation of $\underline{W}$. If these parameters are all equal, then nature simply seeks to maximize the total distance from the mean. In this setting, the optimal action for the seller is to choose stock levels exactly equal to the mean and nature will draw upon both budgets in order to perturb the demand for each item a distance of $\sigma\delta_i^U$ from the mean. However, the policy that was assumed in the calculation of $\underline{W}$ only utilizes a single budget, the budget for small demands.

However, in practice, the policy performs very close to optimal, as we will demonstrate in our numerical experiments. Note that an immediate improvement to the bound, $\overline{W}$, can be made by solving a LP relaxation of a knapsack problem as in (4.8). However, doing so does not provide a closed form solution that can be analyzed easily.

4.4.1 Special Case of Small Lower Bounds

A special case arises when the mean demand for all items is low as in the Air Force setting referenced in Section 2 as well as our second numerical experiment of Section 5.2. In such a setting, the lower bound on ε_i becomes $-\mu_i/\sigma_i$ as we cannot have negative demand quantities. Because the lower bound is constrained in this manner, it can be that $\delta_i^U >> \mu_i/\sigma_i = \left|\delta_i^L\right|$. As a result, we assume that the joint constraint on small demands is no longer active and propose the following modified uncertainty set:

$$U^{'}\left(\boldsymbol{\delta}, \delta_Z^+\right) = \left\{ \boldsymbol{\varepsilon} : \begin{array}{c} \sum_{i=1}^n \varepsilon_i^+ \sigma_i \leq \mathbb{E}\left[\tilde{Z}^+\right] + \delta_Z^+ \sqrt{\mathrm{Var}\left(\tilde{Z}^+\right)} \\ 0 \leq \varepsilon_i^+ \leq \delta_i^U \ \forall\ i \\ 0 \leq \varepsilon_i^- \leq \frac{\mu_i}{\sigma_i} \ \forall\ i \\ \varepsilon_i = \varepsilon_i^+ - \varepsilon_i^- \ \forall\ i \end{array} \right\}. \tag{4.30}$$

Then under this uncertainty set, the following is the mathematical program for the seller:

$$\min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} \sum_{i=1}^n \max\left\{h_i(s_i - (\mu_i + \varepsilon_i\sigma_i)), b_i((\mu_i + \varepsilon_i\sigma_i) - s_i)\right\}$$

subject to

$$\sum_{i=1}^n \varepsilon_i^+ \sigma_i \leq C^+ = \mathbb{E}\left[\tilde{Z}^+\right] + \delta_Z^+ \sqrt{\mathrm{Var}\left(\tilde{Z}^+\right)}, \tag{1}$$

$$0 \leq \varepsilon_i^+ \leq \delta_i^U \ \forall\ i. \tag{2}$$

$$0 \leq \varepsilon_i^- \leq \frac{\mu_i}{\sigma_i} \ \forall\ i \tag{3}$$

$$\varepsilon_i = \varepsilon_i^+ - \varepsilon_i^- \ \forall\ i. \tag{4}$$

We relax constraint (1) and associate a Lagrange multiplier of λ. Then the following is the Lagrangian relaxation of the original problem:

$$
\begin{aligned}
q(\lambda) = \min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} \sum_{i=1}^{n} & \left[\max\left\{ h_i(s_i - (\mu_i + \varepsilon_i\sigma_i)), b_i((\mu_i + \varepsilon_i\sigma_i) - s_i) \right\} \right] \\
& +\lambda\left(C^+ - \textstyle\sum_{i=1}^n \sigma_i\varepsilon_i^+\right) \\
\text{subject to} & \\
& 0 \le \varepsilon_i^+ \le \delta_i^U \ \forall\, i, \\
& 0 \le \varepsilon_i^- \le \tfrac{\mu_i}{\sigma_i} \ \forall\, i, \\
& \varepsilon_i = \varepsilon_i^+ - \varepsilon_i^- \ \forall\, i.
\end{aligned}
$$

Distributing the Lagrange multiplier, incorporating the resulting terms into the maximization, and substituting $\varepsilon_i = \varepsilon_i^+ - \varepsilon_i^-$, we see that

$$
q(\lambda) = \min_{\boldsymbol{s}} \max_{\boldsymbol{\varepsilon}} \sum_{i=1}^{n} \left[\max\left\{ \begin{array}{l} h_i(s_i - \mu_i) + h_i\varepsilon_i^-\sigma_i - (h_i + \lambda)\varepsilon_i^+\sigma_i, \\ b_i(\mu_i - s_i) + (b_i - \lambda)\varepsilon_i^+\sigma_i - b_i\varepsilon_i^-\sigma_i \end{array} \right\} \right] + \lambda C^+.
$$

Examining the coefficients of $\varepsilon_i^+, \varepsilon_i^-$ on both sides of the inner maximization, we obtain the following simplification through the use of indicator variables

$$
q(\lambda) = \min_{\boldsymbol{s}} \sum_{i=1}^{n} \left[\max\left\{ \begin{array}{l} h_i(s_i - \mu_i) + h_i\mu_i, \\ b_i(\mu_i - s_i) + (b_i - \lambda)\delta_i^U\sigma_i\mathbb{1}_{\{b_i > \lambda\}} \end{array} \right\} \right] + \lambda C^+.
$$

Setting the two terms in the maximization equal, we obtain the following stock level

$$
s_i = \mu_i + \frac{b_i - \lambda}{b_i + h_i}\delta_i^U\sigma_i\mathbb{1}_{\{b_i > \lambda\}} - \frac{h_i}{b_i + h_i}\mu_i.
$$

Substituting this inventory level back into the objective function, we obtain

$$
q(\lambda) = \sum_{i=1}^{n} \left[h_i \left(\mu_i + \frac{b_i - \lambda}{b_i + h_i}\sigma_i\delta_i^U\mathbb{1}_{\{b_i > \lambda\}} - \frac{h_i}{b_i + h_i}\mu_i \right) \right] + \lambda C^+.
$$

However, as before, we note that due to our assumption on the budget, C^+,

$$
C^+ \ge \sum_{i=1}^{n} \frac{1}{2}\sigma_i\delta_i^U > \sum_{i=1}^{n} \frac{h_i}{b_i + h_i}\sigma_i\delta_i^U,
$$

and as a result, $\lambda = 0$ minimizes the Lagrangian objective function and has corresponding stock level and cost:

$$s_i = \mu_i + \frac{b_i}{b_i + h_i}\sigma_i\delta_i^U - \frac{h_i}{b_i + h_i}\mu_i,$$
$$q(0) = \sum_{i=1}^{n} \frac{b_i h_i}{b_i + h_i}\mu_i + \frac{b_i h_i}{b_i + h_i}\sigma_i\delta_i^U.$$

Having proven Proposition 4.8, we now use the proposition to lower bound the true optimal cost to the seller from below by $\sum_{i=1}^n h_i \left[\frac{b_i\delta_i^U - h_i\delta_i^L}{b_i+h_i}\sigma_i + \sigma_i\delta_i^L\right]$. Then we are able to analyze the performance of the Lagrangian policy.

Theorem 4.2 *Under the assumptions*

1. $C^+ \geq \max_{|S|=\lceil n/2\rceil} \sum_{i\in S} \sigma_i\delta_i^U$,

2. $C^- \geq \sum_{i\in S} \sigma_i\delta_i^L$,

3. $\min_i b_i > \max_i h_i$, *and*

4. $\delta_i^U \geq \delta_i^L\ \forall\ i$,

the Lagrangian relaxation policy is optimal.

Proof. Observe that because of our assumption that the joint constraint on small demands is non-active, $C^- > \sum_{i=1}^n \sigma_i\delta_i^L$. Therefore, it is always feasible (but potentially suboptimal) for nature to apply minimal demands to each item for a total incremental cost of $\sum_{i=1}^n \left|h_i\sigma_i\delta_i^L\right| = \sum_{i=1}^n h_i\mu_i$. This results in a lower bound on the true cost of

$$\begin{aligned}
&\sum_{i=1}^{n} \left[h_i\left(\mu_i + \frac{b_i}{b_i + h_i}\sigma_i\delta_i^U - \frac{h_i}{b_i + h_i}\mu_i - \mu_i\right) + h_i\mu_i\right] \\
=&\sum_{i=1}^{n} \left[\frac{b_i h_i}{b_i + h_i}\mu_i + \frac{b_i h_i}{b_i + h_i}\sigma_i\delta_i^U\right],
\end{aligned}$$

which is the minimal inventory cost and the minimal incremental cost imposed by nature. However, this is exactly the cost associated with the policy suggested by the best Lagrangian relaxation with $\lambda = 0$. Therefore, in this setting, the Lagrangian relaxation achieves the lower bound and yields the optimal solution.

Recall that the analysis in the previous section led to a $2-$approximation for the general problem. However, in the next section, we perform some numerical experiments which indicate that the policy performs significantly better than this bound, typically coming within a few percentage points of the optimal cost.

5 Numerical Results

While the analysis of the Lagrangian policy in the previous section showed that by using the policy, the seller's resulting cost is at most a factor of 2 larger than the cost under the optimal policy. However, the Lagrangian policy it may perform quite well on average. In fact, in our experiments, the average performance is within 1% of the optimal value. In this section, we perform two numerical studies to evaluate this policy. Numerical Study 1 uses random samples to set the parameters of the problem, and Numerical Study 2 uses data collected from the F-15 fighter aircraft. Recall that the number of extreme points associated with the problem grows combinatorially as a function of the number of items n. In the worst case, there can be as many as $2^n \cdot n!$ possible extreme points where the factor of 2^n corresponds to whether each item experiences backorder or holding costs and an additional factor of $n!$ corresponds to the order in which the budgets of uncertainty are utilized on each item. As a result, in our experiments, we constrain the size of the problem to 5 items, which allows us to efficiently solve each problem to optimality.

5.1 Randomized Parameters

In this section of the numerical study, we simulate random problem instances by choosing parameters uniformly at random over the intervals specified below. This does not necessarily correspond to any real-world setting, but we believe that these problem instances cover a wide range of possible input parameters, and thus are useful as a way to study the performance of our proposed policy.

The demand parameters of the problem were chosen as follows:

$$\mu_i \overset{iid}{\sim} Uniform(0,2)$$

$$\sigma_i \overset{iid}{\sim} Uniform(0,4)$$

$$\delta_i^L \sim \min\{Uniform(0,4), \mu_i/\sigma_i\}$$

$$\delta_i^U \sim (1+X_i)\delta_i^L, \text{ where } X \overset{iid}{\sim} Uniform(0,1).$$

Note that this ensures that $\delta_i^U \geq \delta_i^L$, as we have assumed, and on average, $\sigma_i > \mu_i$. Next, define $\alpha = \frac{\mathbb{E}[b_i]}{\mathbb{E}[h_i]}$. Then the cost parameters of the problem are chosen as follows:

$$h_i \overset{iid}{\sim} Uniform(0,2)$$

$b_i \overset{iid}{\sim} 2\left(\alpha - 2\frac{n}{n+1}\right)X_i + \max_i h_i$, where $X \overset{iid}{\sim} Uniform(0,1)$ and n is the number of items.

Note that $\mathbb{E}[\max_i h_i] = 2\frac{n}{n+1}$. As a result, $\mathbb{E}[b_i] = \alpha$ while ensuring that $\min_i b_i \geq \max_i h_i$ as we have assumed. The positive budget of uncertainty for nature was then chosen uniformly at random between $\max_{Q:|Q|=3} \sum_{i\in Q} \sigma_i\delta_i^U$ and $\sum_{i=1}^5 \sigma_i\delta_i^U$ in order to satisfy the previous assumption. Likewise, the negative budget of uncertainty for nature was then chosen uniformly at random between $\max_{Q:|Q|=3} \sum_{i\in Q} \sigma_i\delta_i^L$ and $\sum_{i=1}^5 \sigma_i\delta_i^L$ in order to satisfy our previous assumption regarding the budgets of uncertainty.

We simulated $10,000$ random problem instances as described above for $\alpha = 5, 10, 50, 100$ which we feel accurately represents the range of backorder cost to holding cost ratios found in real-world problems. For each problem instance, we computed the optimal Lagrange multipliers to determine a set of stock levels. To solve each instance to optimality, we utilized gradient descent by enumerating the set of possible extreme points for nature. We computed the performance of the Lagrangian policy relative to the optimal policy in each instance, recording the average and worst performance ratios which are reported in Table 1 along with 90% confidence intervals for the average performance.

	$\mathbb{E}[b]$	5	10	50	100
Lagrangian	Average	1.0022	1.0037	1.0020	1.0010
		± 5.800e-05	± 7.561e-05	± 5.094e-05	± 3.251e-05
	Worst	1.0261	1.0291	1.0302	1.0239

Table 1: Experimental results for Lagrangian policy for the setting with randomized problem parameters.

The graphic below depicts the performance of the Lagrangian policy for each instance relative to the average performance for 1000 of the simulated instances, with the performance of each iteration in blue and the average performance in red.

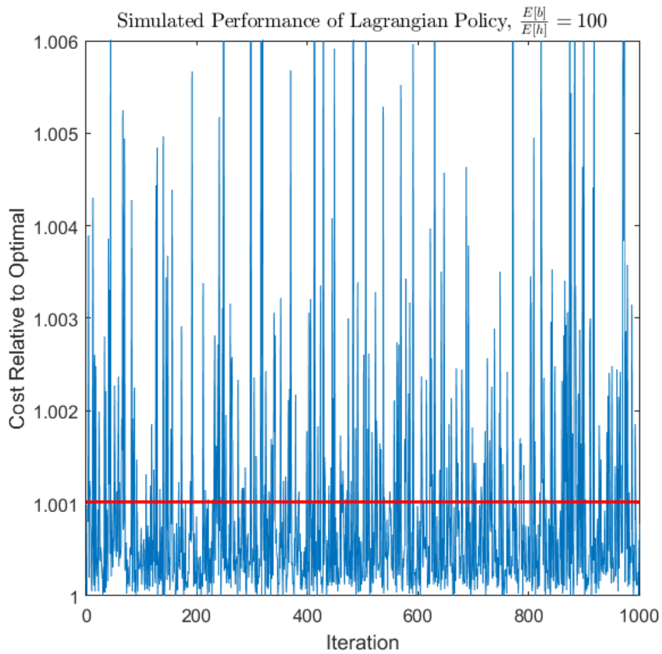

Figure 10: Depiction of the performance of the Lagrangian policy against simulated data when $\alpha = 100$. True performance in blue, average performance in red. Cost and demand parameters chosen as stated above.

We next plot the empirical CDFs for the performance of our randomized experiments in Figure 11. This figure compares the performance under the above values of $\alpha = \mathbb{E}[b_i]/\mathbb{E}[h_i]$. Note that when α is large, while the percentage of instances in which the Lagrangian policy is optimal decreases, the average performance improves greatly. That is to say, when α is small, a high percentage of instances result in an optimal Lagrangian policy, but the right tail is fairly heavy. As α becomes large, fewer instances result in an optimal Lagrangian policy, but the right tail becomes quite light.

However, note that this relationship does not hold for very small values of α. Note that the performance of the policy for $\alpha = 5$ seems to dominate the performance for when $\alpha = 10$. Therefore, the experiments suggest that while the policy performs well for all values of α, the policy performs especially well for very small and very large values of α

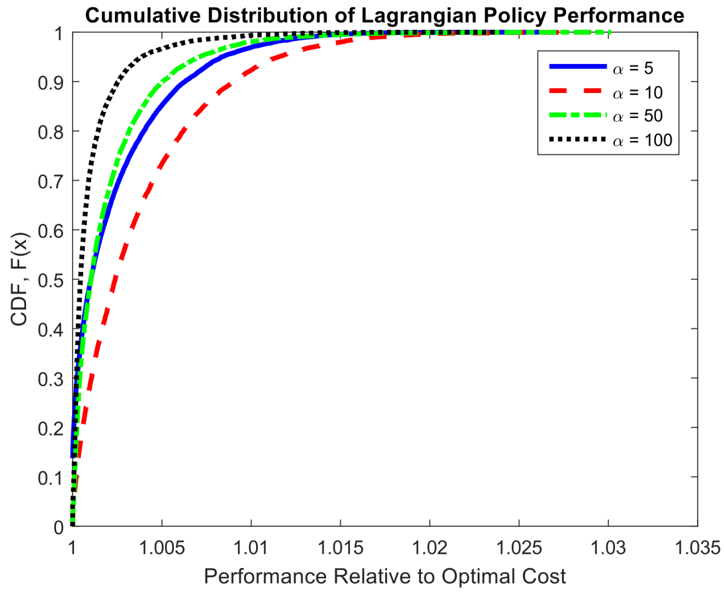

Figure 11: Depiction of the performance of the Lagrangian policy against simulated data for a variety of α.

Observe that the Lagrangian policy performs extremely well on average, with an average cost no larger than 1% more than the optimal solution. The results also suggest that as the ratio of backorder to holding costs becomes large, the performance of the Lagrangian policy approaches the performance of optimal policy, though it may not converge to the optimal policy. Recall that if $\min_i b_i > (n-1)\max_i h_i$, then the approach presented in the beginning of Section 4.3 provides a closed form solution for the optimal policy, and thus may outperform the Lagrangian policy as the ratio of backorder to holding cost becomes large. We also see that the worst case is significantly smaller than the bound indicated by our performance guarantee, with a cost that never exceeds 12% above the optimal cost. Much of this gap between actual performance and the worst case performance guarantee can be explained through the looseness of our proposed bound, $\underline{W}$. As discussed earlier, the 2-approximation was proven using a suboptimal response by nature. The bound can immediately be tightened by utilizing a more sophisticated strategy for nature such as the proposed knapsack method stated in equation (4.8) at the beginning of Section 4.3.

We further examined the problem instances where the Lagrangian policy performed worse than average, but were unable to find a particular structure over the inputs which led to poor performance. More specifically, we fixed certain sets of parameters to be identical while varying the remaining parameters and were still able to find instances where the Lagrangian policy did not perform well for every set of fixed parameters. As an example, we were able to find poorly performing instances after we set the demand parameters $\mu_i, \sigma_i, \delta_i^U, \delta_i^L$ equal across items while randomly generating the cost parameters. Then, when we fixed the cost parameters to be equal across items while varying the demand parameters, we were still able to find poorly performing problem instances.

In the subsequent section, we perform a similar numerical study based on data collected by the United States Air Force for the F-15 fighter aircraft. We feel that the data from a real inventory system will have more structure than the randomly generated instances in this study, and the results of the F-15 fighter aircraft study are significantly improved, demonstrating some form of regularity that was not present in Numerical Study 1.

5.2 F-15 data

In addition to performing a numerical study using simulated data, we also conducted a numerical study utilizing data from the United States Air Force. We use the mean removal rates per 100 flying hours as well as the per-unit cost for 24 Line Replaceable Units (LRUs) found on the F-15 fighter aircraft. These electronic modules are used for communication, radar, etc. and are critical to the ability to operate the aircraft. In this setting, h_i represents the imputed holding cost for each item as described in Section 2. We also assume an identically high backorder cost for each unit, b. This corresponds to the setting in which a backorder results in a non-operational aircraft. We consider a five item subset of the 24 items. These items represents the range of attributes found in the system. The item characteristics from the data are found in Table 2 below.

	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
Removal Rate, μ_i	0.0571	0.0862	0.2222	0.0059	0.0117
Imputed Holding Cost, h_i	3.8	4.9	22.6	1.7	1.4

Table 2: Removal rates and unit costs for a subset of LRUs found on F-15 fighter aircraft.

We consider a scenario in which there are 5 aircraft deployed to an operating base with a one day resupply time. As a result, the maximum number of failures resulting in backorders would be 5 for each item. However, we do not wish to overprotect relative to the mean, so we limit the maximum value of δ_i^U to be 2, representing 2 standard deviations above the mean daily demand. This value of 2 is a realistic value for the operational setting of interest. Clearly, the number of failures is lower bounded by 0. Thus the demand parameters δ_i^U, δ_i^L were set to be:

$$\delta_i^U = \min\left\{2, \frac{5-\mu_i}{\sigma_i}\right\} \ \forall\ i, \qquad\qquad \delta_i^L = -\frac{\mu_i}{\sigma_i} \ \forall\ i.$$

Note that this satisfies our assumption that $\delta_i^U \geq \delta_i^L \ \forall\ i$.

We then examined a few cases where for each item i, σ_i: 1) $\sigma_i = \mu_i$, 2) $\sigma_i = \mu_i + \sqrt{\mu_i}$, and 3) $\sigma_i = \mu_i + 1/\mu_i$. Each of these scenarios ensure that the standard deviation is at least as large as the mean. In practice, σ_i is close to μ_i. Hence, the larger values for σ_i were tested to determine how well the proposed methodology would perform in extreme settings. Additionally, the identically high backorder cost was varied ($b_i = 200, 400, 600, 800 \ \forall\ i$) in order to understand the effect of the backorder cost on the performance of our policy. The budgets of uncertainty, C^+ and C^-, were generated uniformly at random over the intervals $\left[\max_{Q:|Q|=3} \sum_{i\in Q} \sigma_i \delta_i^U, \sum_{i=1}^5 \sigma_i \delta_i^U\right)$ and $\left[\max_{Q:|Q|=3} \sum_{i\in Q} \sigma_i \delta_i^L, \sum_{i=1}^5 \sigma_i \delta_i^L\right)$, respectively. The lower end of these intervals can be thought of as imposing a constraint on the total number of failures of all types. This corresponds to our operational environment; while the number of failures per day may be large, it cannot be the case that every part on every aircraft has failed. The upper end of these intervals corresponds to the case where the budget of uncertainty is large enough that nature can consider the demand for each item separately. Budgets near this value are too conservative for the operational setting that we consider, but can serve as a useful benchmark for the performance of our policy.

We performed 10,000 trials for each of the twelve cases. The average performances of our policy are reported in Table 3 along with 95% confidence interval half-widths. The worst case performances and performance guarantees are reported in Table 4.

$\sigma_i =$	$b = 200$	$b = 400$	$b = 600$	$b = 800$
μ_i	$1.0006 \pm 8.13\text{e-}06$	$1.0003 \pm 5.89\text{e-}06$	$1.0002 \pm 5.48\text{e-}06$	$1.0002 \pm 8.15\text{e-}06$
$\mu_i + \sqrt{\mu_i}$	$1.0002 \pm 3.13\text{e-}06$	$1.0001 \pm 2.27\text{e-}06$	$1.0001 \pm 2.13\text{e-}06$	$1.0001 \pm 1.88\text{e-}06$
$\mu_i + 1/\mu_i$	$1.0001 \pm 7.96\text{e-}07$	$1.0000 \pm 5.78\text{e-}06$	$1.0000 \pm 5.55\text{e-}06$	$1.0000 \pm 4.72\text{e-}06$

Table 3: Average performance of policy in numerical experiments utilizing F-15 fighter aircraft data.

From the table above, we see that the policy performs well on average, with an average performance of no more than one percent greater than the optimal cost. Using the analysis from Theorem 4.1, we compare these values to the worst case approximation guarantees of each case:

$\sigma_i =$	$b = 200$	$b = 400$	$b = 600$	$b = 800$
μ_i	1.0014 **(1.2054)**	1.0010 **(1.2027)**	1.0007 **(1.2018)**	1.0006 **(1.2014)**
$\mu_i + \sqrt{\mu_i}$	1.0006 **(1.0760)**	1.0004 **(1.0751)**	1.0003 **(1.0748)**	1.0002 **(1.0746)**
$\mu_i + 1/\mu_i$	1.0001 **(1.0233)**	1.0001 **(1.0230)**	1.0001 **(1.0229)**	1.0001 **(1.0229)**

Table 4: Worst case performance ratios and approximation guarantees for numerical experiments utilizing F-15 fighter aircraft. Numbers reported as Worst Case (Approximation Guarantee).

Additionally, we plot the empirical CDF for the case where $\sigma_i = \mu_i \ \forall\ i$ below. The two remaining cases yielded nearly identical CDFs and thus have been omitted.

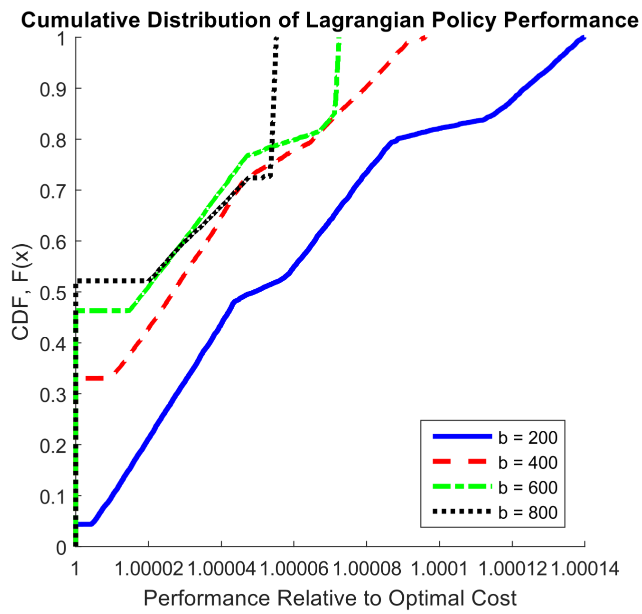

Figure 12: Depiction of the performance of the Lagrangian policy against F-15 data when $\sigma_i = \mu_i \ \forall \ i$.

Observe that the average and worst performance of the policy for the examples perform very well for all three instances. In particular, for all cases where $\sigma_i = \mu_i$ and where $\sigma_i = \mu_i + \sqrt{\mu_i}$, the average and worst costs are better expected as seen in Numerical Study 1 given the ratio of backorder to holding costs. Note that in all cases, the performance under Numerical Study 2 greatly outperforms the performance observed in Numerical Study 1. We believe that this indicates a form of regularity in these real data that was not present in our simulated data.

To summarize the results of our two numerical experiments, the average performance of the policy is much better than the worst case ratio found in the analysis, for a widely varying range of item dependent costs and item dependent demand parameters.

6 Conclusion

In this paper we considered several single-location, single-period robust newsvendor problems with n items, beginning with the simplest case in which all parameters were identical across items. However, as we demonstrated through several example cases, our robust formulation is quite flexible and allows for many modifications, such as non-identical demand and uncertainty parameters across all items as well as non-identical budgets of uncertainty. Notably, the number of constraints governing the uncertainty set grows linearly in the number of items.

We were able to analytically solve several cases for the two-item problem, but closed form solutions do not exist for the general n-item problem as the number of extreme points for nature grew too quickly. As a result, the main contribution of our paper is the approximation algorithm that we propose for the general n-item problem. To implement our algorithm, we only require a $O(n)$ search for the optimal choice of Lagrange multiplier which can then be used to calculate closed form ordering quantities for our robust inventory problem. As a result of the simplicity of implementing this policy, we believe that our robust ordering policy can be easily and successfully utilized in practice. The only inputs needed by our model are the bounds on individual item demand as well as total deviation, which can be easily gathered from historical data or made via managerial decision.

An appealing property of our model is that while asymmetry of either the demand process or the uncertainty sets may present problems in some previous models, our model is able to accommodate various forms of asymmetry in the demand process and joint constraints on nature without any additional computational complexity. As a result, we think that our model can be used successfully to capture very general demands.

We also present very encouraging computational results that confirm the value of the approximation algorithm and indicate that the average performance is very close to optimal.

In particular, while our algorithm does very well on problem instances based on synthetic data, our algorithm has particularly strong performance on problem instances based on real data. In our experiment results, the average performance of our algorithm was well within 1% of the optimal solution.

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